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APPROXIMATION BY PSEUDO-LINEAR DISCRETE OPERATORS

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Abstract. In this note, we construct a pseudo-linear kind discrete operator based on the continuous and nondecreasing generator function. Then, we obtain an approximation to uniformly continuous functions through this new operator. Furthermore, we calculate the error estimation of this approach with a modulus of continuity based on a generator function. The obtained results are supported by visualizing with an explicit example. Finally, we investigate the relation between discrete operators and generalized sampling series.

1. **Introduction.** Discrete operators are a notable research area in approximation theory, and many significant approximation results have been obtained in various forms (see [2, 3, 4, 6, 15, 16, 17, 18, 19, 20, 38]). These operators are connected with generalized sampling series. On the other hand, Bede et al. (see [13]) defined a new type of nonlinear operator by using three types of ordered semirings. Max-product and max-min operators, two of these operators, are frequently studied in approximation and fuzzy theory (see [11, 21, 22, 23, 24, 25, 26, 27, 29, 30, 31, 32, 33, 34, 35, 36, 37, 39, 41, 43]), although the approximation properties of generated pseudo-linear operators are examined a few in the literature (see [12, 13, 14]). It should also observe that the generated pseudo-linear operators may perform better results (see [14]) than the max-product, max-min and linear counterparts.

In this paper, our primary motivation is to introduce pseudo-linear discrete operators utilizing generalized metric spaces based on the pseudo operations with the nondecreasing and continuous generator function g. Then, we examine their approximation results in uniform $g-$distance. Moreover, we investigate the error estimation of this $g-$uniformly convergence. Also, our results are confirmed by graphical illustrations with a suitable discrete kernel. Lastly, we will give an approximation theorem by making use of the relationship between discrete operators and sampling series.

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Since Bede et al. changed the structure of summation and multiplication with pseudo-addition and pseudo-multiplication [13], we need some definitions and notations of this concept given below.

Let g be a function such that $g:\mathbb{R}_0^+ \to \mathbb{R}_0^+$ is continuous and nondecreasing (generator) function with $g(0)=0$. Then, pseudo operations are defined as

$$x \oplus y := g^{-1}(g(x)+g(y))$$

and

$$x \odot y := g^{-1}(g(x).g(y)),$$

where $\mathbf{1} = g^{-1}(1)$ denotes the unit element of the $\odot$ multiplication.

As in [13], for our main approximation theorem we need the following metric structure d_g over $\mathbb{R}_0^+$ given as

$$d_g(u,v) := g^{-1}(|g(u)-g(v)|).$$

Then $d_g:\mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ obeys the following properties (see [13]):

(i) $d_g(x,y) \geq 0$ for all $x,y \in \mathbb{R}_0^+$ and $d_g(x,y)=0$ if and only if $x=y$,
(ii) $d_g(x,y) = d_g(y,x)$ for all $x,y \in \mathbb{R}_0^+$,
(iii) $d_g(x,z) \leq d_g(x,y) \oplus d_g(y,z)$ for all $x,y,z \in \mathbb{R}_0^+$

and we note that, $(\mathbb{R}_0^+, d_g)$ is a generalized metric space (see [40, 42]).

Moreover, instead of usual norm, we consider uniform g-distance, defined as

$$D_g(f,h) = \sup_{x\in\mathbb{R}} d_g(f(x),h(x))$$

for all $f,h:\mathbb{R}\to\mathbb{R}_0^+$.

Now, we deal with the following terms given as follows:

- We denote the space of all uniformly continuous functions (in the Euclidean sense) $f:\mathbb{R}\to\mathbb{R}_0^+$ by $C(\mathbb{R},\mathbb{R}_0^+)$,
- An operator $T:C(\mathbb{R},\mathbb{R}_0^+)\to C(\mathbb{R},\mathbb{R}_0^+)$ is called "continuous in the uniform g-distance" if for all $\varepsilon>0$, there exists a $\delta>0$ such that $D_g(T(f),T(h))<\varepsilon$ whenever $D_g(f,h)<\delta$ (see [13]).
- In addition, a sequence $(f_w)_{w>0}$ is said to be convergent to f in the uniform g-distance, if for all $\varepsilon>0$, there can be found a number w_0 satisfying that $D_g(f_w,f)<\varepsilon$ for all $w\geq w_0$ and is denoted by

$$\lim_{w\to\infty} f_w = f \text{ (g-uniformly).}$$

We also consider the g-modulus of continuity, which is defined by $\omega_g(f,\cdot):\mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that

$$\omega_g(f,\delta) = \sup\{d_g(f(u),f(v)) : u,v\in\mathbb{R},\ |u-v|\leq\delta\},$$

where $f:\mathbb{R}\to\mathbb{R}_0^+$ is bounded in the Euclidean distance. Some important features of g-modulus of continuity are already given below.

Theorem 1.1. *Let f be a bounded and positive function on the whole real line and $\delta,\delta_1,\delta_2>0$. The following statements are valid for the g-modulus of continuity ([13]):*

i) $d_g(f(u),f(v)) \leq \omega_g(f,|u-v|)$ *for all* $u,v\in\mathbb{R}$;
ii) $\omega_g(f,\delta)$ *is nondecreasing with respect to* δ;
iii) $\omega_g(f,0)=0$;
iv) $\omega_g(f,\delta_1+\delta_2) \leq \omega_g(f,\delta_1)\oplus\omega_g(f,\delta_2)$ *for all* $\delta_1,\delta_2\in[0,\infty)$;
v) $\omega_g(f,n\delta) \leq g^{-1}(n)\odot\omega_g(f,\delta)$ *for all* $\delta\in\mathbb{R}_0^+$ *and* $n\in\mathbb{N}$;

vi) $\omega_g(f,\lambda\delta) \leq g^{-1}(\lambda+1) \odot \omega_g(f,\delta)$ *for all* $\delta, \lambda \in \mathbb{R}_0^+$;

vii) $\omega_g(f,\delta) = g^{-1}(\omega(gf,\delta))$ *for all* δ *(Here,* ω *denotes the classical modulus of continuity);*

viii) $\lim_{\delta\to 0} \omega_g(f,\delta) = 0$ *if and only if* f *is bounded and uniformly continuous in the Euclidean sense.*

2. **Pseudo-linear discrete operators.** In this section, we give our main approximation theorem using pseudo-linear discrete operators. For this, we take into account the following discrete operators

$$\sum_{k\in\mathbb{Z}} p_{k,w} f\left(x - \frac{k}{w}\right),$$

which are introduced and studied in [3, 4, 6], where the discrete kernel actualize the following conditions

- $\sum\limits_{k\in\mathbb{Z}} |p_{k,w}| \leq A$ for all $w > 0$.
- $\sum\limits_{k\in\mathbb{Z}} p_{k,w} = 1$ for all $w > 0$.
- There exists a number $r > 0$ such that $\lim\limits_{w\to\infty} \sum\limits_{|k|\geq r} |p_{k,w}| = 0$.

These conditions are known in the literature as approximate identities. One can find or construct many different kernels satisfying the above conditions (see [3, 4, 6]).

We remind that Bede et al. in [13], change the structure of $\sum$ and $\cdot$ by $\oplus$ and $\odot$ respectively and construct pseudo-linear operators with continuous kernels as follows

$$P_n(f;x) = \bigoplus_{i=0}^{n} K_n(x,x_i) \odot f(x_i),$$

where f,$K_n(\cdot,x_i) : X \to [0,+\infty)$, $i = 0,\cdots,n$ are continuous functions and $x_i \in X$ are fixed sampled points. Starting from this point of view, our motivation is to construct pseudo-linear operators with discrete kernels as follows

$$T_w(f;x) = \bigoplus_{k\in\mathbb{Z}} p_{k,w} \odot f\left(x - \frac{k}{w}\right) \qquad (w > 0), \tag{1}$$

where $f : \mathbb{R} \to \mathbb{R}_0^+$ is bounded in the Euclidean sense. Here $\{p_{k,w}\}_{k\in\mathbb{Z},w>0}$ is a family of discrete kernels for all $w > 0$ such that $p_{k,w} > 0$ for all $k \in \mathbb{Z}$, $w > 0$.

For these type of operators, we need some usual conditions given by

$p_1)$ $\bigoplus_{k\in\mathbb{Z}} p_{k,w} = \mathbf{1}$ for all $w > 0$,

$p_2)$ there exists a $r > 0$ such that

$$\lim_{w\to\infty} \bigoplus_{|k|\geq r} p_{k,w} = 0,$$

which are natural in this setting (see [3, 4, 6]). Before giving our approximation theorem, we mention about some features of pseudo-operations and pseudo-linear operators with the following lemmas and propositions.

Lemma 2.1. *For all* $k \in \mathbb{Z}$ *and* $x_k, y_k \in \mathbb{R}_0^+$ *we have*

$$d_g\left(\bigoplus_{k\in\mathbb{Z}} x_k, \bigoplus_{k\in\mathbb{Z}} y_k\right) \leq \bigoplus_{k\in\mathbb{Z}} d_g(x_k,y_k).$$

Proof. Similar to the proof of Lemma 14 in [13]. □

Lemma 2.2. *If $a, b, c \in \mathbb{R}_0^+$, then*

$$d_g(a \odot b, a \odot c) = a \odot d_g(b \odot c)$$

holds [13].

Proposition 1. *Let $f, h \in C(\mathbb{R}, \mathbb{R}_0^+)$. Then the operator T_w in (1) is pseudo-linear in the sense that*

$$T_w\left((\alpha \odot f) \bigoplus (\beta \odot h); x\right) = (\alpha \odot T_w(f; x)) \bigoplus (\beta \odot T_w(h; x))$$

for any $\alpha, \beta \in \mathbb{R}_0^+$.

Proposition 2. *If p_1 holds, then T_w is continuous in the uniform g-distance for all $w > 0$.*

Proof. Let $x \in \mathbb{R}$ and $f, h \in C(\mathbb{R}, \mathbb{R}_0^+)$. Then from Lemma 2.1 and Lemma 2.2 there holds

$$\begin{aligned} d_g(T_w(f; x), T_w(h; x)) &= d_g\left(\bigoplus_{k \in \mathbb{Z}} p_{k,w} \odot f\left(x - \frac{k}{w}\right), \bigoplus_{k \in \mathbb{Z}} p_{k,w} \odot h\left(x - \frac{k}{w}\right)\right) \\ &\leq \bigoplus_{k \in \mathbb{Z}} d_g\left(p_{k,w} \odot f\left(x - \frac{k}{w}\right), p_{k,w} \odot h\left(x - \frac{k}{w}\right)\right) \\ &= \bigoplus_{k \in \mathbb{Z}} p_{k,w} \odot d_g\left(f\left(x - \frac{k}{w}\right), h\left(x - \frac{k}{w}\right)\right) \\ &\leq \bigoplus_{k \in \mathbb{Z}} p_{k,w} \odot D_g(f, h). \end{aligned}$$

Now, using p_1 and the monotonocity of the pseudo-multiplication, the proof is completed. □

Our approximation theorem is obtained as below.

Theorem 2.3. *Let $f : \mathbb{R} \to \mathbb{R}_0^+$ be a uniformly continuous and bounded function in the usual sense. If $p_{k,w}$ satisfies p_1 and p_2, then there holds*

$$\lim_{w \to \infty} T_w(f) = f \quad (g\text{-uniformly}).$$

Proof. Using property (iii) of d_g, we obtain

$$\begin{aligned} d_g(T_w(f; x), f(x)) =& d_g\left(\bigoplus_{k \in \mathbb{Z}} p_{k,w} \odot f\left(x - \frac{k}{w}\right), f(x)\right) \\ \leq& d_g\left(\bigoplus_{k \in \mathbb{Z}} p_{k,w} \odot f\left(x - \frac{k}{w}\right), \bigoplus_{k \in \mathbb{Z}} p_{k,w} \odot f(x)\right) \\ &\oplus d_g\left(\bigoplus_{k \in \mathbb{Z}} p_{k,w} \odot f(x), f(x)\right). \end{aligned} \tag{2}$$

Now, from Lemma 2.1 and Lemma 2.2, we get

$$\begin{aligned} d_g\left(T_w\left(f;x\right),f\left(x\right)\right) \leq & \bigoplus_{k\in\mathbb{Z}} p_{k,w}\odot d_g\left(f\left(x-\frac{k}{w}\right),f\left(x\right)\right) \\ & \oplus d_g\left(\bigoplus_{k\in\mathbb{Z}} p_{k,w}\odot f\left(x\right),f\left(x\right)\right). \end{aligned}$$

Now, from the assumption p_1 and (i), one can easily see that

$$\begin{aligned} d_g\left(\bigoplus_{k\in\mathbb{Z}} p_{k,w}\odot f\left(x\right),f\left(x\right)\right) &= d_g\left(\mathbf{1}\odot f\left(x\right),f\left(x\right)\right) \\ &= d_g\left(f\left(x\right),f\left(x\right)\right) \\ &= 0. \end{aligned}$$

Notice that $g(0)=0$ is the neutral element of pseudo-addition. So we have

$$d_g\left(T_w\left(f;x\right),f\left(x\right)\right) \leq \bigoplus_{k\in\mathbb{Z}} p_{k,w}\odot d_g\left(f\left(x-\frac{k}{w}\right),f\left(x\right)\right).$$

Now, let us divide pseudo-addition into two parts as follows,

$$\begin{aligned} d_g\left(T_w\left(f;x\right),f\left(x\right)\right) \leq & \bigoplus_{|k|<\bar{r}} p_{k,w}\odot d_g\left(f\left(x-\frac{k}{w}\right),f\left(x\right)\right) \\ & \oplus \bigoplus_{|k|\geq\bar{r}} p_{k,w}\odot d_g\left(f\left(x-\frac{k}{w}\right),f\left(x\right)\right) \\ =: & A_1\oplus A_2, \end{aligned}$$

where $\bar{r}$ corresponds to given r in p_2. Since f and g are continuous and f is bounded, then we may find a number $M>0$ such that $d_g\left(f\left(x-\frac{k}{w}\right),f\left(x\right)\right)\leq M$ for all $x\in\mathbb{R}_0^+$, $k\in\mathbb{Z}$ and $w>0$. Hence, since $\odot$ is nondecreasing, it follows from the pseudo-linearity of our operator that

$$A_2 \leq M\odot\bigoplus_{|k|\geq\bar{r}} p_{k,w}.$$

Now, directly from p_2, we have

$$A_2 < M\odot\varepsilon = \varepsilon'$$

for sufficiently large $w>0$.

On the other hand, in A_1, from Theorem 1.1 (i) and (ii), we get

$$d_g\left(f\left(x-\frac{k}{w}\right),f\left(x\right)\right) \leq \omega_g\left(f,\frac{k}{w}\right) \leq \omega_g\left(f,\frac{\bar{r}}{w}\right).$$

Since f is continuous in the usual sense, we observe from $(viii)$ in Theorem 1.1 that

$$\omega_g\left(f,\frac{\bar{r}}{w}\right) < \varepsilon'$$

for sufficiently large $w>0$. Therefore

$$A_1 < \bigoplus_{|k|<\bar{r}} p_{k,w}\odot\varepsilon' \leq \mathbf{1}\odot\varepsilon' = \varepsilon'$$

holds. Finally, taking supremum over $x\in\mathbb{R}$ in (2), we complete the proof. □

3. **Order of approximation.** In this section, we examine the order of convergence by means of g-modulus of continuity. To this end, as in [5, 6, 7, 8, 9, 10, 34], we take into account the following definitions.

Let (u_w) and (v_w) be a sequence of positive real numbers. Then we write

$$u_w = O(v_w) \text{ as } w \to \infty,$$

if one can find the numbers w_0, $C > 0$ satisfying that $u_w \leq C v_w$ for all $w \geq w_0$.

We also define the following Lipschitz class of continuous and bounded functions for $\alpha \in (0, 1]$ as follows:

$$Lip_g^\alpha(\mathbb{R}) = \left\{ f \in C\left(\mathbb{R}, \mathbb{R}_0^+\right) : d_g(f(x), f(y)) \leq L|x-y|^\alpha \text{ for all } x, y \in \mathbb{R} \right\}$$

Then, we get following error estimation.

Theorem 3.1. *Let (u_w) be a null sequence of positive real numbers and $f : \mathbb{R} \to \mathbb{R}_0^+$ be a uniformly continuous and bounded function in the usual sense. If p_1 and the following condition*

$$\bigoplus_{|k| \geq r} p_{k,w} = O(u_w) \text{ as } w \to \infty \tag{3}$$

hold, then we have

$$D_g(T_w(f), f) \leq \omega_g\left(f, \frac{\bar{r}}{w}\right) \bigoplus (M \odot C u_w)$$

for some $C > 0$. Furthermore, if $f \in Lip_g^\alpha(\mathbb{R})$ and $u_w = O(1/w^\alpha)$, then we obtain

$$D_g(T_w(f), f) = (M \oplus 1) \bigodot O\left(\frac{1}{w^\alpha}\right) \text{ as } w \to \infty .$$

Proof. From the proof of Theorem 2.3, it results the following inequality

$$\begin{aligned} D_g(T_w(f; \cdot), f(\cdot)) \leq & \bigoplus_{|k| < \bar{r}} p_{k,w} \odot \omega_g\left(f, \frac{\bar{r}}{w}\right) \\ & \oplus (M \odot \bigoplus_{|k| \geq \bar{r}} p_{k,w}) \\ \leq & \omega_g\left(f, \frac{\bar{r}}{w}\right) \bigoplus (M \odot \bigoplus_{|k| \geq \bar{r}} p_{k,w}). \end{aligned}$$

Then from the assumption (3), there exists a $C > 0$ such that

$$D_g(T_w(f; \cdot), f(\cdot)) \leq \omega_g\left(f, \frac{\bar{r}}{w}\right) \bigoplus (M \odot C u_w),$$

which proves the first part of the theorem. Moreover, if we suppose $f \in Lip_g(\mathbb{R})$ and $u_w = O(1/w^\alpha)$, then one can find an $L > 0$ such that $\omega_g\left(f, \frac{\bar{r}}{w}\right) \leq L\bar{r}/w^\alpha$ and

$$\begin{aligned} D_g(T_w(f; \cdot), f(\cdot)) &\leq L\frac{\bar{r}}{w^\alpha} \bigoplus \left(M \odot \frac{\tilde{C}}{w^\alpha}\right) \\ &\leq \frac{D}{w^\alpha} \bigoplus \left(M \odot \frac{D}{w^\alpha}\right) \end{aligned}$$

for some $\tilde{C}>0$, where $D=\max\{L\bar{r},\tilde{C}\}$. Now, considering $a\bigoplus(b\odot a)=a\bigodot(b\oplus 1)$, we finally have

$$\begin{aligned} D_g\left(T_w\left(f;\cdot\right),f\left(\cdot\right)\right) &\leq (M\oplus 1)\bigodot\frac{D}{w^{\alpha}} \\ &= (M\oplus 1)\bigodot O\left(\frac{1}{w^{\alpha}}\right) \text{ as } w\to\infty. \end{aligned}$$

□

4. **Applications.** In this section, we give some specific kernels to exemplify our approximation theorem. Consider the following cases:

Let $p_{k,w}$ be defined by

$$p_{k,w}=\frac{\sqrt{4^w-1}}{\sqrt{4^w+1}}.\frac{1}{2^{w|k|}}$$

and assume that the generator function $g:\mathbb{R}_0^+\to\mathbb{R}_0^+$, $g\left(x\right)=x^2$. Then there holds

$$\begin{aligned} a\oplus b &= \sqrt{a^2+b^2} \\ a\odot b &= a.b \end{aligned}$$

Now, using these statements we prove that our discrete kernel $p_{k,w}$ satisfies p_1 and p_2.

- For p_1, by the definition of pseudo-addition,

$$\begin{aligned} \bigoplus_{k\in\mathbb{Z}} p_{k,w} &= \sqrt{\sum_{k\in\mathbb{Z}}\frac{4^w-1}{4^w+1}.\frac{1}{4^{w|k|}}} \\ &= \sqrt{\frac{4^w-1}{4^w+1}}\sqrt{\sum_{k\in\mathbb{Z}}\frac{1}{4^{w|k|}}} \\ &= \sqrt{\frac{4^w-1}{4^w+1}}\sqrt{\frac{2}{1-\frac{1}{4^w}}-1} \\ &= 1 \end{aligned}$$

- On the other hand, taking $r=1$

$$\begin{aligned} \bigoplus_{|k|\geq 1} p_{k,w} &= \sqrt{\sum_{|k|\geq 1}\frac{4^w-1}{4^w+1}.\frac{1}{4^{w|k|}}} \\ &= \sqrt{\frac{4^w-1}{4^w+1}}\sqrt{\frac{2}{4^w-1}} \\ &\leq \sqrt{\frac{2}{4^w-1}} \\ &\leq \frac{1}{\sqrt{w}}, \end{aligned} \tag{4}$$

  which proves p_2 and (3) for $u_w=1/w$.

Thus, using the above kernel our operator is convergent to any uniformly continuous and bounded function $f:\mathbb{R}\to\mathbb{R}_0^+$ in the uniform g-distance.

Here, taking

$$f\left(x\right)=\sin x+1 \tag{5}$$

and $p_{k,w}$ as given above, we obtain Figure 1 for certain values of $w>0$.

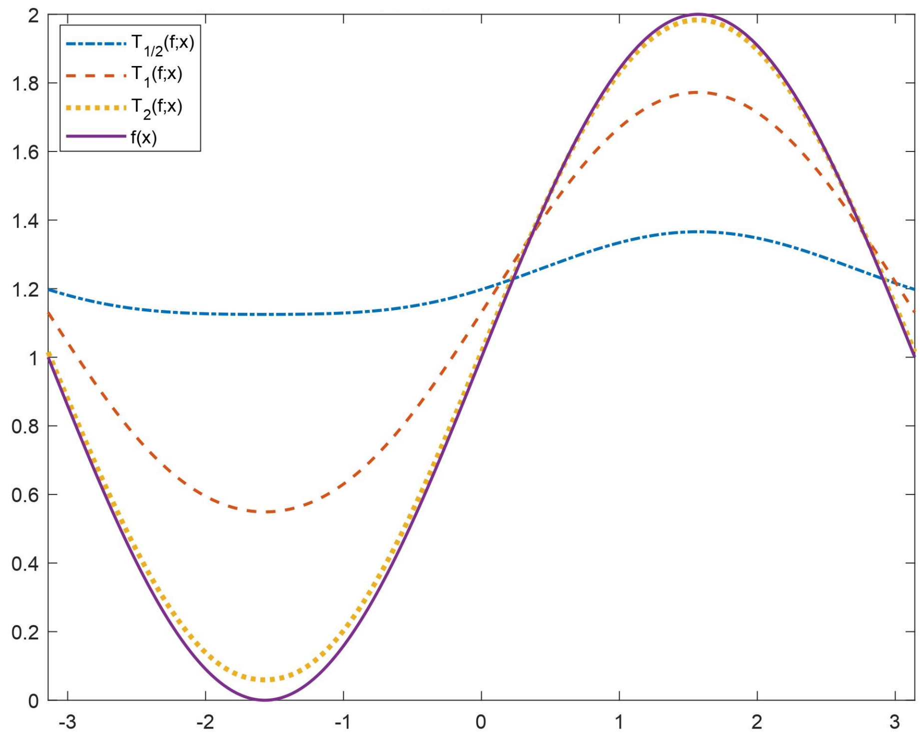

FIGURE 1. Approximation to f given in (5) by pseudo-linear discrete operators

Besides, the given function $f \in Lip_g^{\frac{1}{2}}(\mathbb{R})$, indeed we show as follows:

$$\begin{aligned} d_g\left(f(u), f(v)\right) =& g^{-1}\left(|g(f(u)) - g(f(v))|\right) \\ =& \sqrt{|(1+\sin u)^2 - (1+\sin v)^2|} \\ =& \sqrt{|\sin u - \sin v|}.\sqrt{|2+\sin u + \sin v|} \\ \leq& 2\sqrt{|u-v|}. \end{aligned}$$

Now, using the fact that

$$\begin{aligned} d_g\left(f\left(x-\frac{k}{w}\right), f(x)\right) =& g^{-1}\left(\left|g\left(f\left(x-\frac{k}{w}\right)\right) - g(f(x))\right|\right) \\ \leq& g^{-1}\left(|2g(f(x))|\right) = \sqrt{2.4} = 2\sqrt{2} = M \end{aligned}$$

and considering (4), we finally have

$$\begin{aligned} D_g(T_w(f), f) \leq& (2\sqrt{2} \oplus 1) \bigodot O\left(\frac{1}{\sqrt{w}}\right) \\ =& O\left(\frac{1}{\sqrt{w}}\right). \end{aligned}$$

5. **Further results.** In this part, we indicate that operator (1), in some special cases, turns out to be pseudo-linear generalized sampling series. We should remark

that these type of operators have many applications in image processing, signal processing and etc. [1, 2, 18, 20, 17, 28, 38].

Now, assume that $p_{k,w}$ does not depend on w, namely, $p_{k,w} = \chi(k)$, where $\chi : \mathbb{R} \to \mathbb{R}_0^+$. Then, we obtain the following pseudo linear operator

$$\breve{T}_w(f;x) = \bigoplus_{k\in\mathbb{Z}} \chi(k) \odot f\left(x - \frac{k}{w}\right). \tag{6}$$

Notice that if we write (6) explicitly, we get

$$\breve{T}_w(f;x) = g^{-1}\left(\sum_{k\in\mathbb{Z}} (g\circ\chi)(k) . (g\circ f)\left(x - \frac{k}{w}\right)\right).$$

Considering Lemma 4.2 and Theorem 4.4 in [3] in the above operator, we understand that

$$\breve{T}_w = S_w, \tag{7}$$

where

$$S_w(f;x) = \bigoplus_{k\in\mathbb{Z}} \chi(wx - k) \odot f\left(\frac{k}{w}\right).$$

Then, our operator turns to pseudo-linear sampling operator for all $f : \mathbb{R} \to \mathbb{R}_0^+$ such that $g\circ f \in B^1_{\pi\hat{w}}(\mathbb{R})$ for some $\hat{w} > 0$ and $g\circ\chi \in B^\infty_\pi(\mathbb{R})$.

We recall that the Paley-Wiener space $B^p_T(\mathbb{R})$ $(1 \le p \le \infty)$ is defined by

$$B^p_T(\mathbb{R}) = \big\{ f : \mathbb{R} \to \mathbb{R}_0^+ : f \in L^p(\mathbb{R}) \text{ and } f \text{ has an extension to the whole } \mathbb{C} \\ \text{such that } |f(z)| \le e^{T|z|}\|f\| \text{ for all } z \in \mathbb{C} \big\},$$

where $\|f\|$ is the usual supremum norm of f.

On the other hand, condition p_1 reduces to the following one

$\breve{p}_1)$ $\bigoplus_{k\in\mathbb{Z}} \chi(k) = \mathbf{1}$,

where, condition p_2 is clearly not satisfied. This condition is enough to verify the following theorem.

Theorem 5.1. *Assume that $f : \mathbb{R} \to \mathbb{R}_0^+$ is bounded and uniformly continuous with $g\circ f \in B^1_{\pi\hat{w}}(\mathbb{R})$ for some $\hat{w} > 0$. If $g\circ\chi \in B^\infty_\pi(\mathbb{R})$ and $\breve{p}_1$ is satisfied, then we get*

$$\lim_{w\to\infty} S_w(f) = f \quad (g\text{-uniformly}).$$

Proof. From (7), we may write

$$\begin{aligned} d_g\left(\breve{T}_w(f;x), f(x)\right) &= d_g\left(\bigoplus_{k\in\mathbb{Z}} \chi(k) \odot f\left(x - \frac{k}{w}\right), f(x)\right) \\ &\le d_g\left(\bigoplus_{k\in\mathbb{Z}} \chi(k) \odot f\left(x - \frac{k}{w}\right), \bigoplus_{k\in\mathbb{Z}} \chi(k) \odot f(x)\right) \\ &\oplus d_g\left(\bigoplus_{k\in\mathbb{Z}} \chi(k) \odot f(x), f(x)\right), \end{aligned}$$

where

$$d_g\left(\bigoplus_{k\in\mathbb{Z}} \chi(k) \odot f(x), f(x)\right) = 0$$

from $\breve{p}_1$ and pseudo-linearity of the operator (6). Therefore, we have

$$d_g\left(\breve{T}_w(f;x), f(x)\right) \leq d_g\left(\bigoplus_{k\in\mathbb{Z}} \chi(k) \odot f\left(x-\frac{k}{w}\right), \bigoplus_{k\in\mathbb{Z}} \chi(k) \odot f(x)\right).$$

Now considering Lemma 2.1 and Lemma 2.2 above, we obtain

$$d_g\left(\breve{T}_w(f;x), f(x)\right) \leq \bigoplus_{k\in\mathbb{Z}} \chi(k) \odot d_g\left(f\left(x-\frac{k}{w}\right), f(x)\right). \tag{8}$$

On the other hand, for an arbitrary $\varepsilon > 0$, one can find a $r > 0$ such that

$$\bigoplus_{|k|>r} \chi(k) = g^{-1}\left(\sum_{|k|>r} g(\chi(k))\right) < \varepsilon.$$

Divide the pseudo-addition in (8) as follows

$$d_g\left(\breve{T}_w(f;x), f(x)\right) \leq \bigoplus_{|k|>r} \chi(k) \odot d_g\left(f\left(x-\frac{k}{w}\right), f(x)\right) \oplus \bigoplus_{|k|\leq r} \chi(k) \odot d_g\left(f\left(x-\frac{k}{w}\right), f(x)\right).$$

For the first part, from the boundedness of f we get

$$\bigoplus_{|k|>r} \chi(k) \odot d_g\left(f\left(x-\frac{k}{w}\right), f(x)\right) \leq M \odot \bigoplus_{|k|>r} \chi(k) < M \odot \varepsilon = \varepsilon'.$$

For the second part, using similar arguments in A_1 in the proof of Theorem 2.3

$$\bigoplus_{|k|\leq r} \chi(k) \odot d_g\left(f\left(x-\frac{k}{w}\right), f(x)\right) < \varepsilon'$$

holds for sufficiently large $w > 0$, which completes the proof. □

Modifying the Fejer kernel, we can easily find a suitable kernel satisfying the new conditions. Let χ be defined by

$$\chi(x) = \frac{\frac{1}{4} sinc^4\left(\frac{x}{2}\right)}{\left(\sum_{k\in\mathbb{Z}} \frac{1}{2} sinc^2\left(\frac{k}{2}\right)\right)^2}, \tag{9}$$

where

$$sinc(x) = \begin{cases} \frac{\sin x}{x}; & \text{if } x \neq 0 \\ 0; & \text{if } x = 0. \end{cases}$$

Now assuming $g(x)=\sqrt{x}$, we observe that

$$\begin{aligned}\bigoplus_{k\in\mathbb{Z}}\chi(k) &= g^{-1}\left(\sum_{k\in\mathbb{Z}} g\circ\chi(k)\right)\\ &= g^{-1}\left(\sum_{k\in\mathbb{Z}}\frac{\frac{1}{2}sinc^2\left(\frac{k}{2}\right)}{\sum\limits_{k\in\mathbb{Z}}\frac{1}{2}sinc^2\left(\frac{k}{2}\right)}\right)\\ &= 1\end{aligned}$$

In addition, since

$$g\circ\chi(x)=\frac{\frac{1}{2}sinc^2\left(\frac{x}{2}\right)}{\sum\limits_{k\in\mathbb{Z}}\frac{1}{2}sinc^2\left(\frac{k}{2}\right)},$$

then it is not hard to see from Example 4.5 in [3] that $g\circ\chi\in B_{\pi}^{1}(\mathbb{R})\subset B_{\pi}^{\infty}(\mathbb{R})$. So our new kernel (9) fulfilled all the conditions of Theorem 5.1.

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