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Research article

Data-driven $\mathcal{H}_\infty$ control of constrained systems: An application to bilateral teleoperation system

Ibrahim Kucukdemiral [a], Hakan Yazici [b,*], Bilal Gormus [c], Geraint Paul Bevan [a]

[a] Department of Applied Science, School of Computing, Engineering and Built Environment, Glasgow Caledonian University, Glasgow G4 0BA, UK
[b] Department of Mechanical Engineering, Yildiz Technical University, Istanbul, Turkey
[c] Department of Mechatronics Engineering, Istanbul Gedik University, Istanbul, Turkey

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ABSTRACT

A novel, identification-free, data-driven (DD) $\mathcal{H}_\infty$ control method is presented for discrete-time (DT) linear time-invariant (LTI) systems under physical limitations and norm-bounded disturbances. The presented approach does not demand information on system matrices or any measurements of disturbance affecting the system. The only information needed to develop a static state-feedback (SF) controller is the bounds on disturbances, states and control signals. It is assumed that only the disturbance input matrix and the performance matrices the user generally defines are known, and all others are entirely unknown. The proposed method relies on the closed-loop (CL) parametrization of the LTI system with control input and state measurements. The disturbances affecting the system states are handled as affine uncertainties, later represented as Linear Fractional Transformation (LFT). For obtaining a less conservative controller, a full block S-procedure method (FBSPM) is used, which takes advantage of relaxations such as convex hull relaxation or Pólya relaxation for the inner approximation of the disturbance set with arbitrary precision. Numerical illustrations and extensive case studies on a bilateral teleoperation system indicate that the proposed design method allows us to obtain very effective controllers which never exceed the bounds of the state and input variables and are capable of reference and force tracking.

1. Introduction

The model-based control method is an important design procedure for control theory. In control engineering applications, system models are often partially or fully unknown in advance. Therefore, in practice, system models need to be defined from the measured data using various system identification techniques. This approach entails applying a two-stage control method, whose stages are system identification and controller design. The data-driven controller (DDC) design method offers a solution to this challenge by bypassing the need for model identification and directly using system data to design the controller. This innovative approach has the potential to revolutionize control engineering by simplifying the design process and improving the performance of control systems. In cases where it is difficult or impossible to use system identification methods, data-driven-based control techniques can offer extremely effective solutions [1]. In a DD control method, the information that is directly held by the measured data are used in the design of the controller with a predetermined fixed structure. As is known in the adaptive control (AC) method, controller design can be done without using a model of the system. However, the main difference between DD control and AC is that parameter setting relies on using large data stacks instead of a single input–output sample or several samples [2].

DD control theory started with the pioneering work of Ziegler–Nichols [3] and then led to the development of data-based control methods such as the direct AC [4] and the artificial neural network methods [5]. Recently, many studies have been carried out to handle the optimal control problem using measured data. Skelton and Shi developed a data-based linear quadratic Gaussian (LQG) controller using only knowledge of the plant input–output data. They proposed a batch form Riccati equation method to obtain a data-based optimal control law without using any information of the plant [6]. Then, Aangenent et al., applied an optimal controller design having Markovian system parameters which consist of a data-based observer [7]. The state feedback control gain is not calculated directly in the DD linear quadratic (LQ) controller design proposed in these two studies. Instead, the optimal control gain is estimated directly from the data for each time instant. Therefore, these studies did not solve the

* Corresponding author.
E-mail addresses: ibrahim.kucukdemiral@gcu.ac.uk (I. Kucukdemiral), hyazici@yildiz.edu.tr (H. Yazici), bilal.gormus@gedik.edu.tr (B. Gormus), geraint.bevan@gcu.ac.uk (G.P. Bevan).

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LQ control problem expressed in the classical formulation. On the other hand, Goncalves et al. presented a new DD optimal controller method to calculate the infinite horizon state feedback linear quadratic regulator (LQR) controller gain using a data stack which consists of input and state of the plant [8]. In the method developed by Willems et al., when an input or noise signal is persistently excited with sufficiently high order, the respective input or noise signals span the entire system trajectories [9]. Recent contributions have built on this result. For instance, a dissipativity-based approach is defined to identify the system from measured data in [10], and a robust controller is presented for discrete-time LTI systems. Another instance by de Persis and Tesi is an application on the parametrization of linear feedback systems to solve feedback stabilization and the LQR problem [11]. This method shows that significant control problems can be solved by using data-dependent linear matrix inequalities (LMIs). By extending [11], a robust controller is designed by employing an input state trajectory of finite length under assumed disturbance bound [12]. This control technique guarantees CL stability and satisfies the $\mathcal{H}_\infty$ performance constraint of an LTI system. In addition, Berberich et al. developed a multi-step robust DD model predictive DDC, which guaranteed the exponential stability of the CL of LTI systems having limited additional output measurement noise [13].

Bilateral teleoperation control is a complex and pressing issue that requires innovative solutions. A typical bilateral teleoperation system involves human and environmental-induced exogenous forces, master and slave subsystems, and communication channels [14]. Environmental-induced exogenous forces only come into play when the slave system interacts with an environment. The control system must simultaneously ensure both the tracking of subsystem trajectories and exogenous forces, if present, while also addressing the inherent tradeoff between force tracking and closed-loop stability [15]. Many different control methods have been proposed to address this challenge [16, 17]. Additionally, system identification in a bilateral teleoperation system with such a complex exogenous-force structure can pose another challenge for data-driven control methods.

As discussed above, papers focusing on LMI-based data-driven controller design are rare. Moreover, the actuator saturation problem still needs to be solved for systems with unknown but bounded dynamics. Therefore, in this paper, we develop a new, less conservative controller synthesis method to guarantee the CL stability of a DD control system using finite-length data generated by an LTI system. The fact that there is still more room is our primary motivation for obtaining an identification-free, DD $\mathcal{H}_\infty$ control for discrete LTI systems subjected to physical constraints and norm-bounded disturbances. The main goals of this paper are described with the following highlights:

- The proposed method is a significant contribution to the field of robust control as it offers a highly flexible and adaptable solution for systems with saturated actuators. Unlike traditional methods, the proposed method does not require any prior knowledge of system matrices or measurements of disturbance forces. This makes it efficient and easy to implement in various systems and situations. All that is needed for designing a static state-feedback controller are bounds on disturbances, states, and control signals. This greatly simplifies the design process and eliminates the need for extensive prior knowledge or data collection. Additionally, the method assumes knowledge of disturbance input and performance output matrices, but no other matrices are needed. Overall, this makes the proposed method a powerful and convenient tool for robust control of systems with saturated actuators.
- The proposed technique relies on the CL parametrization of the LTI system with control input and state measurements. Affine uncertainties in the system describe the disturbances affecting the system states and are later represented as LFT. In the controller design, we employ the FBSPM and the convex-hull relaxation approach for the inner approximation of the disturbance set with arbitrary precision. This allows us to achieve highly accurate and reliable results with a minimal computational burden. However, the method can be easily adapted to other approaches, such as Pólya relaxation or the Sum of Squares, for even greater flexibility.
- The proposed method utilizes a nested attractive ellipsoid method to account for saturated actuators. By incorporating a bounding condition on the CL energy, we can impose limits on the system's disturbances. Additionally, the method allows us to define conditions on the magnitude of the control signal and certain system states. This ability to handle saturation while meeting performance requirements makes the proposed method a valuable tool for robust control of systems with saturated actuators.

The proposed method has been extensively tested and validated using a bilateral teleoperation system. Two different case studies were conducted to evaluate the controller's performance under various bounds on disturbances, states, and control signals. The results of these studies demonstrate the method's effectiveness, with strong performance consistently observed in all cases. These numerical results provide strong evidence of the usefulness and reliability of the proposed method in a wide range of situations.

The structure of the remainder of the paper is as follows: In Section 2, we provide an overview of the necessary background and notation. Section 3 presents the problem formulation in detail. The proposed method for the data-driven control of constrained systems is introduced in Section 4. Section 5 demonstrates the effectiveness of the developed controller through simulation studies. Finally, in Section 6, we summarize our findings and draw conclusions.

2. Preliminaries and notation

Unless otherwise stated, lower italic characters stand for vectors or scalars, and capital letters denote matrices. I_m represents an identity matrix of dimension $m \times m$. $\mathbb{N}$ is used for the set of natural numbers, and $\mathbb{R}$ defines the set of real numbers. Furthermore, $\mathbb{Z}$ will show the set of integers. G^T symbolizes the transpose of the matrix G. $\mathsf{He}\{P\} = P + P^T$ for all $P \in \mathbb{R}^{n\times n}$. The symbol $\star$ represents symmetric off-diagonal blocks in a symmetric matrix. $\mathsf{diag}\{\cdot\}$ shows block diagonal matrices, whereas $\mathsf{conv}\{\mathcal{S}\}$ denotes the convex hull of finite set $\mathcal{S}$. $A \succeq (\preceq) 0$ defines A as a symmetric positive (negative) semi-definite matrix. We shall denote a length-N sequence of state vectors $\{x_k\}_{k=0}^{N-1}$ starting at time instant $k = 0$ by $X = \begin{bmatrix} x_0 & x_1 & \cdots & x_{N-1} \end{bmatrix}$ and its one-step delayed version as $X_+ = \begin{bmatrix} x_1 & x_2 & \cdots & x_N \end{bmatrix}$. Similarly, we define $W = \begin{bmatrix} w_0 & w_1 & \cdots & w_{N-1} \end{bmatrix}$, and $U = \begin{bmatrix} u_0 & u_1 & \cdots & u_{N-1} \end{bmatrix}$. $\mathbf{1}$ stands for a vector whose entries are equal to 1. For two vectors $x, y \in \mathbb{R}^n$, $x \le y$ means that for all $i = 1, \ldots, n$, $x_i \le y_i$. Given a matrix $P \in \mathbb{R}^{n\times n}$, where $P = P^T \succeq 0$, the set $\mathcal{E}(P, r) \triangleq \{x \in \mathbb{R}^n : x^T P x \le r\}$ is called the ellipsoid associated with P.

Definition 1. For a discrete-time linear time-invariant (LTI) system of the form $x_{k+1} = Ax_k + B_u u_k$, a collected input-state data matrix $\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$ is considered "persistently exciting" if it has full row rank. This means that the matrix rows are linearly independent, and the data are sufficiently rich and varied to capture the system's behaviour accurately. Persistently exciting data are necessary for certain control design techniques to ensure that the system's dynamics can be accurately identified and modelled.

Lemma 1 ([9]). *Consider a controllable discrete-time LTI system governed by difference equation $x_{k+1} = Ax_k + B_w w_k + B_u u_k$. If the augmented input data matrix $\left[\begin{smallmatrix} W \\ U \end{smallmatrix}\right]$ has a full row rank, then $\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$ is persistently exciting.*

Lemma 2 (*Projection Lemma [18]*). *Given matrices $\mathcal{U}$ and $\mathcal{V}$ which have independent range spaces, and a symmetric matrix $\mathcal{Z}$, there exists an unstructured matrix $\mathcal{X}$ that satisfies*

$$\mathcal{U}^T \mathcal{X} \mathcal{V} + \mathcal{V}^T \mathcal{X}^T \mathcal{U} + \mathcal{Z} \prec \mathbf{0}, \tag{1}$$

if and only if the following inequalities according to $\mathcal{X}$ are satisfied:

$$\mathcal{N}_{\mathcal{U}}^T \mathcal{Z} \mathcal{N}_{\mathcal{U}} \prec \mathbf{0}, \tag{2}$$

$$\mathcal{N}_{\mathcal{V}}^T \mathcal{Z} \mathcal{N}_{\mathcal{V}} \prec \mathbf{0}, \tag{3}$$

where $\mathcal{N}_{\mathcal{U}}$ and $\mathcal{N}_{\mathcal{V}}$ are any matrices whose columns form a basis of the null spaces of $\mathcal{U}$ and $\mathcal{V}$, respectively.

3. Problem formulation

Consider an augmented discrete LTI system model which contains both the plant dynamics and the filter:

$$\mathcal{H} \left\{ \left[\begin{array}{c} x_{k+1} \\ \hline z_k \end{array} \right] = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C & D_w & D_u \end{array} \right] \left[\begin{array}{c} x_k \\ \hline w_k \\ u_k \end{array} \right], \quad x_0 = 0, \right. \tag{4}$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^{m_w}$ and $u \in \mathbb{R}^{m_u}$ are the state, disturbance and control vectors, respectively. From this point forward, we will drop the explicit time dependence of the signals and use x^+ to represent $x(k+1)$ for simplicity. Unless otherwise stated, a stand-alone signal x will be assumed to represent the value of the signal at time k. We assume that all state vector elements are measurable in real time and that the system is subject to constraints on both the state and control inputs. Additionally, we assume that the disturbances acting on the system are bounded. These conditions are summarized briefly as

$$\begin{aligned} \mathcal{X} &= \{x : Tx \le \mathbf{1}\} \\ \mathcal{U} &= \{u : Hu \le \mathbf{1}\} \\ \mathcal{W} &= \left\{ w \in \mathbb{R}^{m_w} : w_k^T w_k \le \bar{w}^2, \ \forall k \ge 0 \right\} \end{aligned} \tag{5}$$

where $T \in \mathbb{R}^{n_T \times n}$ and $H \in \mathbb{R}^{n_H \times n}$. The exogenous input vector $w \in \mathbb{R}^{m_w}$ represents all external disturbances and measurement noise that can impact the system states and outputs. It is assumed to be generally unknown, except for its bound $\bar{w}$. While the system matrices A and B_u are also assumed to be unknown, all other matrices can be either known or defined by the user with the appropriate dimensions. This allows for flexibility in the model and the design process while coping with a certain level of uncertainty that needs to be accounted for in the control design.

This paper aims to develop an identification-free, entirely DD $\mathcal{H}_\infty$ static full SF controller expressed as $u = Kx$ so that the CL $\|\mathcal{H}\|_\infty$ gain of the system defined by $\sup_{\|w\| \neq 0} \frac{\|z\|_2}{\|w\|_2} \le \gamma$ is minimized, while control and state trajectories never exceed the predefined bounds.

To apply the data-driven control technique presented in this paper, the system must first undergo an open-loop initial operation for a duration of N samples, with the condition $n + m_u < N$ being satisfied. During this initial operation, the system is subjected to random and rich input signal sequences U and artificially generated disturbance signal sequences $W \in \mathscr{W}$, where each column of $W \in \mathbb{R}^{m_w \times N}$ is randomly and uniformly chosen from the ball, $\|w\|_2 \le \bar{w}$. The exact values of W are unknown and may vary during operation. All state signal sequences X are generated from the measured data from 0 to $N-1$. Following the $(N-1)$th sample, the controller is calculated by solving a convex optimization problem and implemented on the system. This process allows us to design a controller tailored to the system's specific characteristics and operating conditions.

4. Main results

4.1. Closed-loop system parametrization from data

Using the SF control law $u_k = Kx_k$ along with (4), one can easily obtain the CL system model as follows:

$$\mathcal{H}_{cl} \left\{ \left[\begin{array}{c} x^+ \\ \hline z \end{array} \right] = \left[\begin{array}{c|c} \mathcal{A}_{cl} & \mathcal{B}_{cl} \\ \hline \mathcal{C}_{cl} & \mathcal{D}_{cl} \end{array} \right] \left[\begin{array}{c} x \\ \hline w \end{array} \right], \quad x_0 = 0, \right. \tag{6}$$

where

$$\mathcal{A}_{cl} = A + B_u K, \ \mathcal{B}_{cl} = B_w, \ \mathcal{C}_{cl} = C + DK, \ \mathcal{D}_{cl} = D_w. \tag{7}$$

As we have no information about the system matrices A and B_u, our next step will be to express all terms involving these matrices in terms of matrices that can be derived from the available input–output data $\{x_k, u_k, w_k\}_{k=0}^{N-1}$. To do this, we will use a lemma that allows us to express these matrices in terms of the data. This will allow us to proceed with the control design process, even without complete knowledge of the system matrices.

Lemma 3 ([12]). *If there exist matrices $G \in \mathbb{R}^{N \times n}$ and $K \in \mathbb{R}^{m_u \times n}$ which satisfy*

$$\begin{bmatrix} I_n \\ K \end{bmatrix} = \begin{bmatrix} X \\ U \end{bmatrix} G, \tag{8}$$

then

$$A + B_u K = (X_+ - B_w W) G$$

where W is an unknown but bounded disturbance signal matrix which satisfies $W \in \mathscr{W}$. Moreover, if $\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$ is persistently exciting and $N \ge n + m$, then for any K there exists G which satisfies (8).

4.2. Control of constrained systems (Model based approach)

In order to achieve efficient control while still satisfying constraints on the control signal and system states, we can use a common Lyapunov matrix. However, this approach can result in conservative controllers. To reduce the conservatism of the control design, we can apply dilated matrix inequality conditions using the Bounded Real Lemma in dilated form. This will be a key component in our control technique.

Lemma 4 ([19]). *For the CL system (6), $\|\mathcal{H}_{cl}\|_\infty < \gamma$ if, and only if, there exists a matrix $Y \in \mathbb{R}^{n \times n}$ such that*

$$\begin{bmatrix} Y + Y^T - Q & \star & \star & \star \\ \mathcal{A}_{cl} Y & Q & \star & \star \\ \mathcal{C}_{cl} Y & 0 & \gamma I & \star \\ 0 & \mathcal{B}_{cl}^T & \mathcal{D}_{cl}^T & \gamma I \end{bmatrix} \succ 0 \tag{9}$$

For the sake of analysis, let us temporarily suppose that all the matrices in (4) are known. Then, given the disturbance bound $\bar{w}$, Lemma 5 can be used to design a static state-feedback controller $u = Kx$ for the system in (4) subject to (5), using dilated matrix inequalities.

Lemma 5. *For a given $\alpha \in (0, 1)$, system (4), subjected to disturbances signals satisfying $w_k^T w_k \le \bar{w}^2$ for all $k > 0$, can be* *made asymptotically stable with a control law $u = LY^{-1}x$, if there exist matrices $Q = Q^T$, Y and L such that*

$$\begin{bmatrix} (1-\alpha)Q & \star & \star \\ \alpha(AY+B_uL) & \alpha(Y+Y^T)-Q & \star \\ 0 & \alpha B_w^T & \alpha I \end{bmatrix} \succ 0, \tag{10}$$

$$\begin{bmatrix} Q & L^TH^T\mathbf{e}_i \\ \mathbf{e}_i^THL & \left(\frac{1}{\bar{w}}\right)^2 \end{bmatrix} \succ 0, \qquad \forall i = 1,\dots,n_H \tag{11}$$

$$\begin{bmatrix} Q & Y^TT^T\mathbf{e}_i \\ \mathbf{e}_i^TTY & \left(\frac{1}{\bar{w}}\right)^2 \end{bmatrix} \succ 0, \quad \forall i = 1,\dots,n_T. \tag{12}$$

Moreover, all constraints in (5) are satisfied.

Proof. Consider a Lyapunov function $V \triangleq x^T\mathcal{P}x$ along the CL system trajectory (6). Let us define the rate-of-change in the system's energy $\mathcal{H}_{cl}$ as $\Delta V \triangleq V_{k+1} - V_k$. Note that for the CL stability, $\Delta V < 0$ needs to be satisfied for all $k > 0$, which is also equivalent to

$$\Delta V = \Delta V + \alpha V - \alpha w^Tw + \alpha w^Tw - \alpha V < 0 \tag{13}$$

for a scalar $\alpha \in (0, 1)$. Defining $r \triangleq \Delta V + \alpha V - \alpha w^Tw$, if $r < 0$ then

$$\Delta V < \alpha\left(w^Tw - V\right) < \alpha\left(\bar{w}^2 - V\right). \tag{14}$$

Note that the inequality condition (14) justifies that the ellipsoid $\mathscr{E}(\mathcal{P}, \bar{w}^2)$ is controlled-invariant. Any trajectory starting within the ellipsoid $\mathscr{E}(\mathcal{P}, \bar{w}^2)$ shall never leave it since ΔV becomes zero on or outside the ellipsoid. Then, choosing $\phi = \begin{bmatrix} x^T & w^T \end{bmatrix}^T$ and using the system definition (6), the invariance condition $r < 0$ is equivalent to

$$\begin{aligned} &\phi^T \begin{bmatrix} \mathcal{A}_{cl}^T \\ \mathcal{B}_{cl}^T \end{bmatrix} \mathcal{P} \begin{bmatrix} \mathcal{A}_{cl} & \mathcal{B}_{cl} \end{bmatrix} \phi - (1-\alpha)\phi^T \begin{bmatrix} I \\ 0 \end{bmatrix} \mathcal{P} \begin{bmatrix} I & 0 \end{bmatrix} \phi \\ &- \alpha\phi^T \begin{bmatrix} 0 \\ I \end{bmatrix} \\ &\times \begin{bmatrix} 0 & I \end{bmatrix} \phi < 0. \end{aligned} \tag{15}$$

In compact form, this can be expressed as

$$\underbrace{\begin{bmatrix} I & 0 \\ \mathcal{A}_{cl} & \mathcal{B}_{cl} \\ 0 & I \end{bmatrix}^T}_{\mathcal{N}_{\mathcal{U}}^T} \underbrace{\begin{bmatrix} (\alpha-1)\mathcal{P} & 0 & 0 \\ 0 & \mathcal{P} & 0 \\ 0 & 0 & -\alpha I \end{bmatrix}}_{\mathcal{Z}} \underbrace{\begin{bmatrix} I & 0 \\ \mathcal{A}_{cl} & \mathcal{B}_{cl} \\ 0 & I \end{bmatrix}}_{\mathcal{N}_{\mathcal{U}}} \prec 0. \tag{16}$$

To derive dilated matrix inequality conditions, we shall employ the method, Extension-IV introduced in [20]. To this end, based on the definition of $\mathcal{N}_{\mathcal{U}}$, one can choose $\mathcal{U} = \begin{bmatrix} \mathcal{A}_{cl} & -I & \mathcal{B}_{cl} \end{bmatrix}$. Similarly, choosing $\mathcal{N}_{\mathcal{V}} = \left[\begin{smallmatrix} I & 0 & 0 \\ 0 & 0 & I \end{smallmatrix}\right]^T$, causes the strict inequalities $\mathcal{P} \succ 0$ and $\alpha \in (0, 1)$, where the former one guarantees that V is a Lyapunov function and the latter is needed for the invariance of the CL energy. Note that this particular choice of $\mathcal{N}_{\mathcal{V}}$ leads to $\mathcal{V} = \begin{bmatrix} 0 & \alpha I & 0 \end{bmatrix}$. Then, with all these ingredients and the selection of $\mathcal{X} = Y^{-1}$, the application of the projection Lemma 1 gives

$$\mathrm{He}\left\{ \begin{bmatrix} \mathcal{A}_{cl}^T \\ -I \\ \mathcal{B}_{cl}^T \end{bmatrix} Y^{-1} \begin{bmatrix} 0 & \alpha I & 0 \end{bmatrix} \right\} + \mathcal{Z} \prec 0. \tag{17}$$

Furthermore, defining $Q \triangleq Y^T\mathcal{P}Y$, $L \triangleq KY$ and application of congruence transformation using $\mathrm{diag}\{Y, Y, I\}$ on (17) along with the definitions of $\mathcal{A}_{cl}$, and $\mathcal{B}_{cl}$, yields (10).

Note that since $\mathcal{P} \succ 0$ and $\alpha \in (0, 1)$, element $(2, 2)$ of (17) implies that $Y^{-1} + Y^{-T} \succ 0$. This implies Y is a full rank matrix and, therefore, invertible.

Since we now have a bounding condition on the CL energy through the matrix inequality condition (10), we are in a position to impose additional constraints on the control gain so that the control signal always satisfies $Hu \le \mathbf{1}$. To this end, let us rewrite the magnitude condition over the channels of the control signal as

$$\mathbf{e}_i^THKx \le 1, \quad \forall i = 1,\dots,n_H \tag{18}$$

where $\mathbf{e}_i$ is a n-dimensional column vector whose ith entry is equal to 1 and all other entries are zero. Then, (18) can be written as

$$x^TK^TH^T\mathbf{e}_i\mathbf{e}_i^THKx \le 1, \quad \forall i = 1,\dots,n_H \tag{19}$$

Note that this is nothing but an ellipsoid $\mathscr{E}(K^TH^T\mathbf{e}_i\mathbf{e}_i^THK, 1)$. Therefore, if

$$\mathcal{E}(\mathcal{P}, \bar{w}^2) \subseteq \mathcal{E}(K^TH^T\mathbf{e}_i\mathbf{e}_i^THK, 1) \tag{20}$$

for all $i = 1,\dots,n_H$ and $t > 0$, along with $r < 0$, then the CL system (6) is asymptotically stable, and the controller will never saturate. Note that (20) is equivalent to

$$\frac{\mathcal{P}}{\bar{w}^2} \succeq K^TH^T\mathbf{e}_i\mathbf{e}_i^THK, \qquad \forall i = 1,\dots,n_H \tag{21}$$

which can be further expressed as

$$\begin{bmatrix} \mathcal{P} & K^TH^T\mathbf{e}_i \\ \star & \left(\frac{1}{\bar{w}}\right)^2 \end{bmatrix} \succ 0, \qquad \forall i = 1,\dots,n_H \tag{22}$$

using the Schur complement formulae [21]. Finally, application of congruence transformation on (22) by using $\mathrm{diag}\{Y, I\}$ and using the definitions $L \triangleq KY$ and $Q \triangleq Y^T\mathcal{P}Y$ gives (11).

In case there are conditions on the magnitude of some states to satisfy such that

$$Tx \le \mathbf{1} \tag{23}$$

then the CL system trajectories need to satisfy $\mathcal{E}(\mathcal{P}, \bar{w}^2) \subseteq \mathcal{E}(T^T\mathbf{e}_i\mathbf{e}_i^TT, 1)$ for all $i = 1,\dots,n_T$ as well. Following a similar procedure outlined in the derivation of input conditions, the constraints on states can be restated as an LMI (12). This concludes the proof.

4.3. Data-driven control design for systems subjected to physical constraints

In this section, we will apply the model-based control design technique from the previous section to the system governed by difference Eq. (4) using the DD approach. To do this, we will use the results from Lemma 3. This lemma relies on the disturbance data matrix $W \in \mathscr{W}$, which is unknown but bounded. Our proposed approach will treat this term as uncertainty within the linear fractional representation framework.

To derive tractable and less conservative DD synthesis conditions, we will use the full block S-procedure [22] to solve our problem. Lemma 6 is crucial for obtaining a DD solution to the problem. However, first we need to define a multiplier matrix Φ that satisfies

$$\begin{bmatrix} W^T \\ I \end{bmatrix}^T \underbrace{\begin{bmatrix} \Psi & \chi \\ \chi^T & \Pi \end{bmatrix}}_{\Phi} \begin{bmatrix} W^T \\ I \end{bmatrix} = W\Psi W^T + \mathrm{He}\{W\chi\} + \Pi \prec 0,$$

$$\forall W \in \mathscr{W}. \tag{24}$$

Additionally, let us introduce the LFT of W and $\mathcal{G}$ as

$$W \star \underbrace{\left[\begin{array}{c|c} \mathcal{G}_{11} & \mathcal{G}_{12} \\ \hline \mathcal{G}_{21} & \mathcal{G}_{22} \end{array}\right]}_{\mathcal{G}} \triangleq \mathcal{G}_{22} + \mathcal{G}_{21}W(I - \mathcal{G}_{11}W)^{-1}\mathcal{G}_{12}, \tag{25}$$

which is called *well-posed* when $(I - \mathcal{G}_{11}W)$ is invertible for all $W \in \mathscr{W}$.

Our results are developed on top of the following Lemma:

Lemma 6 (*[23]*)**.** *The LFT $W \star \mathcal{G}$ is well-posed and*

$$\mathrm{He}\{W \star \mathcal{G}\} \triangleq W \star \mathcal{G} + (W \star \mathcal{G})^T \succ 0, \; W \in \mathscr{W}, \tag{26}$$

holds if, and only if,

$$\begin{bmatrix} \mathcal{G}_{21}\Pi\mathcal{G}_{21}^T + \mathrm{He}\{\mathcal{G}_{22}\} & \mathcal{G}_{21}\Pi\mathcal{G}_{11}^T + \mathcal{G}_{21}\chi^T + \mathcal{G}_{12}^T \\ * & \Psi + \mathcal{G}_{11}\Pi\mathcal{G}_{11}^T + \mathrm{He}\{\mathcal{G}_{11}\chi^T\} \end{bmatrix} \succ 0, \tag{27}$$

and there exists a matrix Φ that satisfies (24).

Now we are ready to present our main result as follows:

Theorem 1. *For a given $\alpha \in (0,1)$ and $\gamma > 0$, the $\mathcal{H}_\infty$ norm of the CL system (6), which is subjected to disturbance signals satisfying $w_k^T w_k \leq \bar{w}$ for all $k > 0$, is less than γ, if there exist matrices Y, M, Π_1, Π_2, χ_1, χ_2, Ψ_1, Ψ_2 of appropriate dimensions that satisfy $XM = Y$, inequalities in (12) and*

$$\begin{bmatrix} W^T \\ I \end{bmatrix}^T \begin{bmatrix} \Psi_i & \chi_i \\ \chi_i^T & \Pi_i \end{bmatrix} \begin{bmatrix} W^T \\ I \end{bmatrix} \prec 0, \qquad i = 1, 2, \tag{28}$$

$$\begin{bmatrix} Y + Y^T - Q & \star & \star & \star & \star \\ X_+M & Q + B_w\Pi_1B_w^T & \star & \star & \star \\ CY + DUM & 0 & \gamma I & \star & \star \\ 0 & B_w^T & D_w^T & \gamma I & 0 \\ M & -\chi_1B_w^T & 0 & 0 & \Psi_1 \end{bmatrix} \succ 0, \tag{29}$$

$$\begin{bmatrix} (1-\alpha)Q & \star & \star & \star \\ \alpha X_+M & \begin{pmatrix} \alpha(Y+Y^T) - Q \\ +B_w\Pi_2B_w^T \end{pmatrix} & \star & \star \\ 0 & \alpha B_w^T & \alpha I & \star \\ \alpha M & -\chi_2B_w^T & 0 & \Psi_2 \end{bmatrix} \succ 0, \tag{30}$$

$$\begin{bmatrix} Q & M^TU^TH^T\mathbf{e}_i \\ \mathbf{e}_i^THUM & \left(\frac{1}{\bar{w}}\right)^2 \end{bmatrix} \succ 0, \qquad \forall\, i = 1, \ldots, n_H \tag{31}$$

Then, a DD, static SF controller can be constructed as $u = UMY^{-1}x$. Moreover, all constraints in (5) are satisfied.

Proof. The proof requires consideration of the matrix inequalities (9) and (10) as they involve disturbance matrix W. Considering $\mathcal{A}_{cl} = (X_+ - B_wW)G$, $K = UG$ and defining $M \triangleq GY$, (9) can be rewritten as

$$\begin{bmatrix} Y + Y^T - Q & \star & \star & \star \\ (X_+ - B_wW)M & Q & \star & \star \\ CY + DUM & 0 & \gamma I & \star \\ 0 & B_w^T & D_w^T & \gamma I \end{bmatrix} \succ 0, \tag{32}$$

which can be further expressed in the LFT framework as

$$\underbrace{\begin{bmatrix} Y + Y^T - Q & \star & \star & \star \\ X_+M & Q & \star & \star \\ CY + DUM & 0 & \gamma I & \star \\ 0 & B_w^T & D_w^T & \gamma I \end{bmatrix}}_{\mathcal{G}_{22}+\mathcal{G}_{22}^T} + \mathrm{He}\left\{ \underbrace{\begin{bmatrix} 0 \\ -B_w \\ 0 \\ 0 \end{bmatrix}}_{\mathcal{G}_{21}} W(I - \underbrace{0}_{\mathcal{G}_{11}} W)^{-1} \underbrace{\begin{bmatrix} M & 0 & 0 & 0 \end{bmatrix}}_{\mathcal{G}_{12}} \right\} \succ 0. \tag{33}$$

Then, application of (27) gives (28) and (29) for $i = 1$.

Similarly, using the same definitions in (10), gives

$$\underbrace{\begin{bmatrix} (1-\alpha)Q & \star & \star \\ \alpha X_+M & \alpha(Y+Y^T) - Q & \star \\ 0 & \alpha B_w^T & \alpha I \end{bmatrix}}_{\mathcal{G}_{22}+\mathcal{G}_{22}^T} + \mathrm{He}\left\{ \underbrace{\begin{bmatrix} 0 \\ -B_w \\ 0 \end{bmatrix}}_{\mathcal{G}_{21}} W(I - \underbrace{0}_{\mathcal{G}_{11}} W)^{-1} \underbrace{\begin{bmatrix} \alpha M & 0 & 0 & 0 \end{bmatrix}}_{\mathcal{G}_{12}} \right\} \succ 0. \tag{34}$$

Then, the application of (27) yields (28) and (30) for $i = 2$. There is no change in the LMI conditions of the state-related bounds; therefore, this concludes the proof.

In summary, the proposed data-driven $\mathcal{H}_\infty$ control can be obtained by solving the following algorithm:

Algorithm 1 (*Design algorithm*)**.** *The controller design can be accomplished by following these steps:*

1. *Select w_{max}, u_{max}, x_{max}, and N, and generate random exogenous forces and control signals that satisfy the constraints for a duration of N samples and $\left[\begin{smallmatrix} W \\ U \end{smallmatrix}\right]$ has full row rank and excite system (4) by defining the input matrix U such that $\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$ is persistently exciting. Record X and X_+*
2. *If $\left[\begin{smallmatrix} W \\ U \end{smallmatrix}\right]$ does not have full row rank, return to Step 1 and regenerate W and U, otherwise proceed with Step 3.*
3. *Use a line search to find the optimal value of α such that the minimum closed-loop $\mathcal{H}_\infty$ gain γ is obtained for the conditions $XM = Y$ and the LMIs in (12), (28), (29), (30), (31).*
4. *If a feasible solution exists, construct the optimal DD state-feedback controller as $u = UMY^{-1}x$.*
5. *If a feasible solution does not exist, the user may relax the bounds on the states and inputs until a feasible solution is found.*

Remark 1. The matrix inequality conditions of Theorem 1 cannot be solved in finite time due to the LMI condition (28), which relies on the exact knowledge of W. Note that (28) has to be satisfied for all admissible values of $W \in \mathscr{W}$. Nevertheless, a feasibility check is needed for an infinite number of LMI conditions. Consequently, one needs to obtain sufficient conditions that are tractable instead of (28). To derive a finite number of LMI conditions that ensure the satisfaction of (24), we need to employ some relaxation methods. For this purpose, Pólya's method or Convex-hull relaxation can be used for polytopic regions. However, using the sum-of-squares (SOS) approach (see [24], and [25] and the references therein) would be more suitable for regions described by polynomial inequalities. In [26], two relaxation methods can be found, namely, Pólya relaxation and convex-hull relaxation [27] to investigate the feasibility of (24) employing a finite number of LMI constraints. In this study, we used the convex-hull approach, which can be summarized as follows.

4.4. Convex-hull relaxation to obtain finite-dimensional LMI conditions for (28)

Note that a polytopic region $\mathcal{W}$ can be described as the convex hull of a given set of finitely many vertex matrices such as $\mathcal{W}_e =$

$\{W^1, \ldots, W^\kappa\}$. where

$$\mathcal{W} = \mathrm{conv}(\mathcal{W}_e) = \left\{ \sum_{j=1}^{\kappa} s_j W^j \text{ subject to } \sum_{j=1}^{\kappa} s_j = 1,\ s_j \geq 0 \right\}. \tag{35}$$

It is shown in [26] that the left-hand side of (28) is convex if and only if Ψ_1 and Ψ_2 are both positive definite matrices. Then one can replace condition (28) with

$$\begin{bmatrix} (W^j)^T \\ I \end{bmatrix}^T \begin{bmatrix} \Psi_i & \chi_i \\ \chi_i^T & \Pi_i \end{bmatrix} \begin{bmatrix} (W^j)^T \\ I \end{bmatrix} \prec 0, \quad \forall\, i = 1, 2, \text{ and } \forall j = 1, \ldots, \kappa. \tag{36}$$

and $\Psi_1 \succeq 0$, $\Psi_2 \succeq 0$ to obtain a convex-hull relaxation.

4.5. Reference tracking control

The disturbance rejection problem can also be turned into a tracking problem by slightly modifying the state description. To that end, we introduce a filtered tracking error signal, e, in (4), which leads to

$$\mathcal{H}_e \left\{ \begin{bmatrix} e_+ \\ x_+ \\ \hline z \end{bmatrix} = \left[\begin{array}{cc|ccc} I & C_1 & D_r & 0 & 0 \\ 0 & A & B_r & B_d & B_u \\ \hline J_1 & J_2 C_1 & J_2 D_r & 0 & 0 \end{array} \right] \begin{bmatrix} e \\ x \\ \hline w_r \\ w_d \\ u \end{bmatrix}, \right.$$
$$x_0 = 0,\ e_0 = 0 \tag{37}$$

where J_1 and J_2 stand for the user defined weighting matrices and C_1 and D_r are the filter matrices in appropriate dimensions. Then taking

$$B_w \leftarrow \begin{bmatrix} D_r & 0 \\ B_r & B_w \end{bmatrix}, \quad C \leftarrow \begin{bmatrix} J_1 & J_2 C_1 \end{bmatrix}, \quad D = 0,$$
$$D_w \leftarrow \begin{bmatrix} J_2 D_r & 0 \end{bmatrix} \tag{38}$$

in Theorem 1, allows us to solve the servo-mechanism (reference tracking) problem by using the DD approach.

5. Simulation studies

In this section, the performance of the developed DD $\mathcal{H}_\infty$ controller is examined in comparison with a 1-DoF bilateral teleoperation system that considers reference and force tracking. All simulations are accomplished in MATLAB and Simulink using the SeDuMi solver [28], and Yalmip [29] parser.

Consider a bilateral teleoperation system described by the continuous-time (CT) linear model in [14]. Note that A and B matrices are needed to simulate the system, but they are not assumed to be known for the controller design. Hence, we shall consider the system dynamics:

$$\dot{x}_0(t) = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -0.1 \end{bmatrix}}_{\bar{A}_{ct}} \underbrace{\begin{bmatrix} x_m(t) \\ v_m(t) \\ x_s(t) \\ v_s(t) \end{bmatrix}}_{x_0(t)} + \underbrace{\begin{bmatrix} 0 \\ 0.1 \\ 0 \\ 0 \end{bmatrix}}_{\bar{B}_{rct}} w_r(t)$$
$$+ \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.1 \end{bmatrix}}_{\bar{B}_{dct}} w_d(t) + \underbrace{\begin{bmatrix} 0 & 0 \\ 0.1 & 0 \\ 0 & 0 \\ 0 & 0.1 \end{bmatrix}}_{\bar{B}_{uct}} \underbrace{\begin{bmatrix} u_m(t) \\ u_s(t) \end{bmatrix}}_{u(t)} \tag{39}$$

Here, the trajectories of the master position and velocity are denoted by x_m and v_m, while x_s and v_s represent the trajectories of the slave position and velocity, respectively. u_m and u_s are the master and slave control inputs, respectively. Finally, the exogenous forces of the master and slave are represented by w_r and w_d, respectively, and they stand for the human operator and environment-induced exogenous forces. Specifically, w_r is composed of the reference signal r_h, and human hand dynamics w_h. Hence,

$$\begin{aligned} w_r(t) &= r_h(t) - \underbrace{(0.1x_m(t) + 0.5v_m(t))}_{w_h(t)}, \\ w_d(t) &= 1x_s(t) + 0.1v_s(t), \end{aligned} \tag{40}$$

where

$$r_h(t) = \begin{cases} 1 & 1 \leq t \leq 30 \\ 0 & \text{otherwise} \end{cases} \tag{41}$$

and t represents time in seconds. The performance objectives of the control system are the reference tracking both in master and slave subsystems and the force tracking between w_r and w_d.

For obtaining a discrete-time system, the CT system is discretized with a sampler having a sampling period $T_s = 0.1s$. Hence, one can obtain

$$x_{0+} = \underbrace{\begin{bmatrix} 1 & 0.0995 & 0 & 0 \\ 0 & 0.99 & 0 & 0 \\ 0 & 0 & 1 & 0.0995 \\ 0 & 0 & 0 & 0.99 \end{bmatrix}}_{\bar{A}} \underbrace{\begin{bmatrix} x_m \\ v_m \\ x_s \\ v_s \end{bmatrix}}_{x_0} + \underbrace{\begin{bmatrix} 0.0005 \\ 0.01 \\ 0 \\ 0 \end{bmatrix}}_{\bar{B}_r} w_r$$
$$+ \underbrace{\begin{bmatrix} 0 \\ 0 \\ -0.0005 \\ -0.01 \end{bmatrix}}_{\bar{B}_d} w_d + \underbrace{\begin{bmatrix} 0.0005 & 0 \\ 0.01 & 0 \\ 0 & 0.0005 \\ 0 & 0.01 \end{bmatrix}}_{\bar{B}_u} \underbrace{\begin{bmatrix} u_m \\ u_s \end{bmatrix}}_{u}. \tag{42}$$

This problem defines a disturbance rejection and needs to be translated into a reference tracking problem. To this end, let us define extra states of tracking errors

$$\underbrace{\begin{bmatrix} e_m \\ e_s \end{bmatrix}}_{e_+} \triangleq \begin{bmatrix} 1 & 0 & C_m & 1 & 0 & 0 \\ 0 & 1 & C_s & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_m \\ e_s \\ x_0 \\ w_r \\ w_d \\ u \end{bmatrix} \tag{43}$$

where $e_m = e_s = 0$. Here, $C_m = \begin{bmatrix} -1 & 0 & 0 & 0 \end{bmatrix}$ and $C_s = \begin{bmatrix} 0 & 0 & -1 & 0 \end{bmatrix}$ are output matrices which identify the variables to track w_r. Finally, the dynamic model of the discrete-time bilateral teleoperation system can be written as follows:

$$\begin{aligned} x_+ &= \underbrace{\begin{bmatrix} 1 & 0 & C_m \\ 0 & 1 & C_s \\ 0_{4\times1} & 0_{4\times1} & \bar{A} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} e_m \\ e_s \\ x_0 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ \bar{B}_r & \bar{B}_d \end{bmatrix}}_{B_w} \underbrace{\begin{bmatrix} w_r \\ w_d \end{bmatrix}}_{w} \\ &\quad + \underbrace{\begin{bmatrix} 0_{1\times2} \\ 0_{1\times2} \\ \bar{B}_u \end{bmatrix}}_{B_u} \underbrace{\begin{bmatrix} u_m \\ u_s \end{bmatrix}}_{u} \\ z &= \underbrace{\begin{bmatrix} \eta_1 & 0 & \eta_2 C_m \\ 0 & \delta_1 & \delta_2 C_s \\ 0 & 0 & 0_{1\times4} \end{bmatrix}}_{C} x + \underbrace{\begin{bmatrix} \eta_2 & 0 \\ \delta_2 & 0 \\ \beta & -\beta \end{bmatrix}}_{D_w} w + \underbrace{\begin{bmatrix} 0_{3\times2} \end{bmatrix}}_{D} u \end{aligned} \tag{44}$$

Table 1
Computation results in case study 1.

Controller	γ	K
Nominal Continuous-Time	0.56	$\begin{bmatrix} -80.2733 & -35.5527 & -39.2890 & -13.0344 & 85.1540 & 52.1119 \\ -37.3547 & -11.9348 & -78.8052 & -34.6051 & 49.9785 & 83.6418 \end{bmatrix}$
Data-Driven	7.49	$\begin{bmatrix} 88.8553 & 38.9532 & -82.4649 & -161.6379 & -102.5816 & -2.3947 \\ -11.1758 & 22.0294 & -10.0396 & -7.2151 & -227.9851 & -110.0837 \end{bmatrix}$

Here, C, D_w and D are performance matrices that define z. While the first and second rows of z allow us to satisfy reference tracking, the last is used for satisfactory force tracking (force reflection) performance. Additionally, non-negative weighting parameters η_1, η_2, δ_1, δ_2 and β are used to provide a trade-off among control outputs, and they satisfy $\eta_1 + \eta_2 + \delta_1 + \delta_2 + \beta = 1$.

Similarly, in the CT setting (39), the reference tracking can be achieved by introducing augmented states associated with the integral of tracking errors, which leads to

$$\underbrace{\begin{bmatrix} e_m(t) \\ e_s(t) \end{bmatrix}}_{e(t)} \triangleq \begin{bmatrix} 0 & 0 & C_m & 1 & 0 & 0 \\ 0 & 0 & C_s & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \int e_m(t) \\ \int e_s(t) \\ x_0(t) \\ w_r(t) \\ w_d(t) \\ u(t) \end{bmatrix} \tag{45}$$

Hence, the CT bilateral teleoperation system model can be obtained as follows:

$$x(t) = \underbrace{\begin{bmatrix} 0 & 0 & C_m \\ 0 & 0 & C_s \\ 0_{4\times 1} & 0_{4\times 1} & \bar{A}_{ct} \end{bmatrix}}_{A_{ct}} \underbrace{\begin{bmatrix} \int e_m(t) \\ \int e_s(t) \\ x_0(t) \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ \bar{B}_{r_{ct}} & \bar{B}_{d_{ct}} \end{bmatrix}}_{B_{w_{ct}}} \underbrace{\begin{bmatrix} w_r(t) \\ w_d(t) \end{bmatrix}}_{w(t)}$$

$$+ \underbrace{\begin{bmatrix} 0_{1\times 2} \\ 0_{1\times 2} \\ \bar{B}_{u_{ct}} \end{bmatrix}}_{B_{u_{ct}}} \underbrace{\begin{bmatrix} u_m(t) \\ u_s(t) \end{bmatrix}}_{u(t)} \tag{46}$$

$$z(t) = \underbrace{\begin{bmatrix} 0 & 0 & \eta C_m \\ 0 & 0 & \delta C_s \\ 0 & 0 & 0_{1\times 4} \end{bmatrix}}_{C_{ct}} x(t) + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \beta & -\beta \end{bmatrix}}_{D_w} w(t) + \underbrace{\begin{bmatrix} 0_{3\times 2} \end{bmatrix}}_{D} u(t)$$

Here, C_{ct} is used for reference tracking performance and non-negative weighting parameters are selected to satisfy $\eta + \delta + \beta = 1$.

To investigate the DD control performance, two different case studies are utilized. The first case includes simulation results of DD and CT nominal controllers, with both having saturated actuators. Moreover, to show that a DD controller can learn effectively in training, as a second case study, a comparison study is carried out using the material in [12].

5.1. Case study 1

Assuming that disturbance, state and control input bounds are $w_{max} = \begin{bmatrix} 1.05 & 1.05 \end{bmatrix}^T$,
$x_{max} = \begin{bmatrix} 1.5 & 1.5 & 1.5 & 1.5 & 1.5 & 1.5 \end{bmatrix}^T$ and
$u_{max} = \begin{bmatrix} 150 & 150 \end{bmatrix}^T$, respectively, the application of Theorem 1 gives the target $\mathcal{H}_\infty$ norm and the associated control gain of the DD controller as shown in Table 1. These computation results are produced for $\alpha = 0.525$, while the number of training samples is selected as $N = 20$. The DD controller's exact value is obtained as $\gamma_t = 7.5$, computed by applying the DD controller to the exact model. Lastly, weighting parameters are chosen as $\eta_1 = 0.175$, $\eta_2 = 0.175$, $\delta_1 = 0.175$, $\delta_1 = 0.175$ and $\beta = 0.30$. To investigate the performance of the proposed DD control method, a CT nominal controller having a saturated actuator is also considered, whose derivation is given in Appendix. In the simulations, the input bounds for the saturated actuator are selected as $u_{max} = \begin{bmatrix} 150 & 150 \end{bmatrix}^T$ and weighting parameters are chosen as $\eta = 0.35$, $\delta = 0.35$ and $\beta = 0.30$. Finally, the CT nominal controller simulation results are shown in Table 1. This table demonstrates that more than eight times higher $\mathcal{H}_\infty$ norm is found with the proposed DDC compared to the nominal CT controller.

The learning performance of the DD control is dramatically affected by N. Our observations show a correlation between the size of the data set, N and the success of approximating the actual $\mathcal{H}_\infty$ gain, γ_t. However, this gap does not decrease after a certain value and converges. In our studies of the Bilateral Teleoperation system, we observed that the value of $N = 20$ was sufficient for a successful and reasonable controller design. However, this value might be different in other physical systems. In the bilateral teleoperation system, it has to be chosen under the condition $N \geq 9 = n + m_u$. To describe the learning performance, errors between γ and γ_t have been considered $((\gamma - \gamma_t)/\gamma_t)$ and shown in the left-hand side of Fig. 1. Note that γ stands for the system's computed $\mathcal{H}_\infty$ gain when the system matrices are unknown. These statistics will allow us to identify how compatible our results are with the exact $\mathcal{H}_\infty$ norm of the closed-loop system. Moreover, the parameter α is used to obtain the minimum value of γ through a line search. The variation of γ for different values of α, is shown in the right-hand side of Fig. 1. Note that learning performance may change in every experiment. Therefore, approximately similar solutions have to be considered for successful learning performance. In light of that observation, every percentage error and the computed γ value have been averaged after 5 experiments, and the results are shown in Fig. 1. As can be seen from Fig. 1, the DD control shows the best learning performance approximately after 20 samples of training. Here, γ_t can be calculated with only 3.17% error with the proposed method. In Case Study 1, we used a solution that finds γ_t exactly. On the other hand, α is chosen as 0.525 to get the minimum γ. Note that all values of $\alpha > 0.71$ and $\alpha < 0.18$ generate infeasible solutions.

Fig. 2 shows the reference tracking performance of the proposed DDC. The trajectories of x_m and x_s indicate that the DD controller can effectively follow the reference despite physical state constraints. However, it exhibits a slightly higher overshoot than the CT nominal controller. Despite this, the DD controller produces a highly satisfactory transient and steady-state response.

In Fig. 3, the force-tracking performances of the same controllers are demonstrated. Both controllers achieve similar performance in tracking exogenous forces in the nominal case. Moreover, examining the results of the proposed DDC, we conclude that, despite some overshoot, exogenous forces stay within

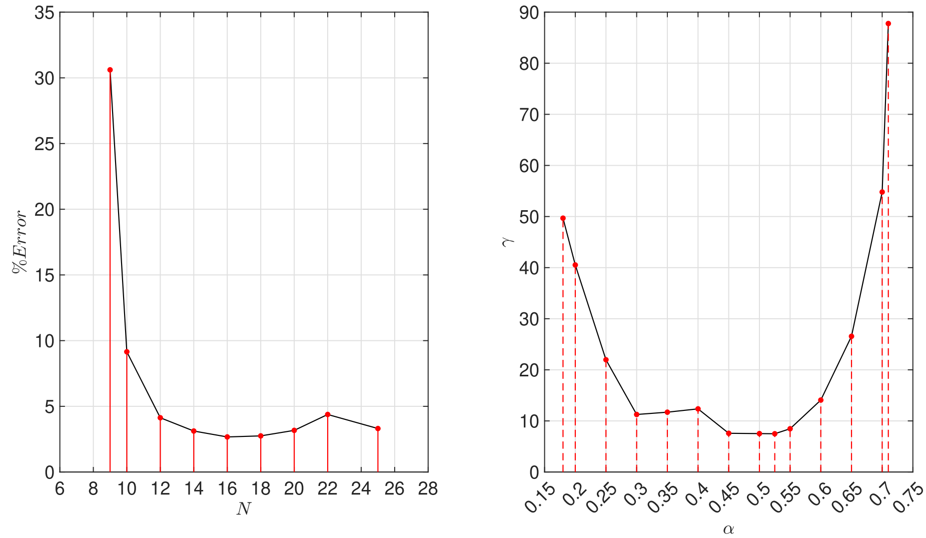

Fig. 1. Determination of optimum values of N and α.

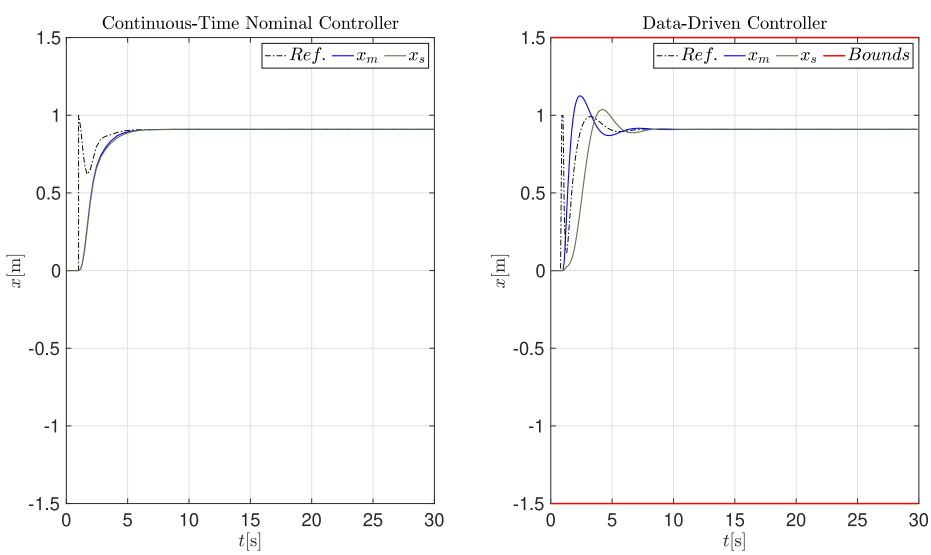

Fig. 2. Reference tracking performance for Case Study 1.

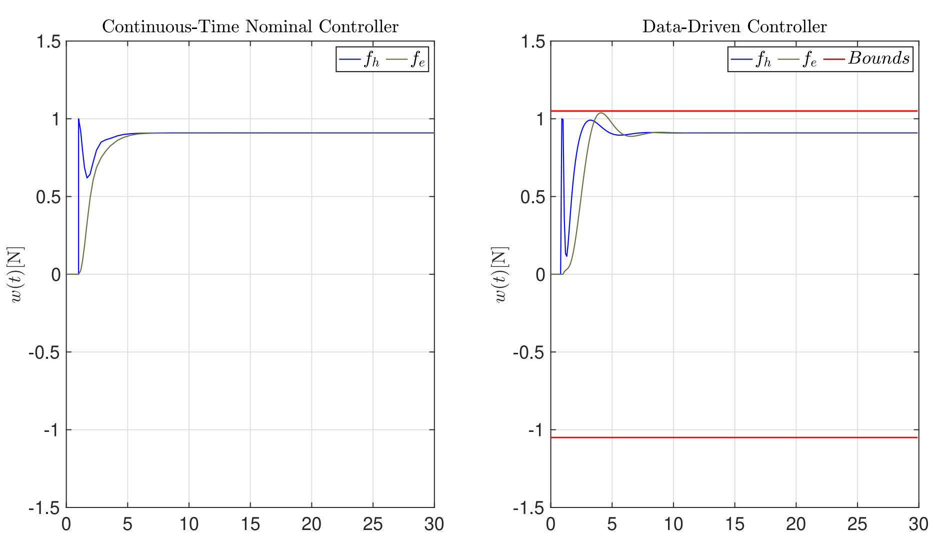

Fig. 3. Force tracking performance for Case Study 1.

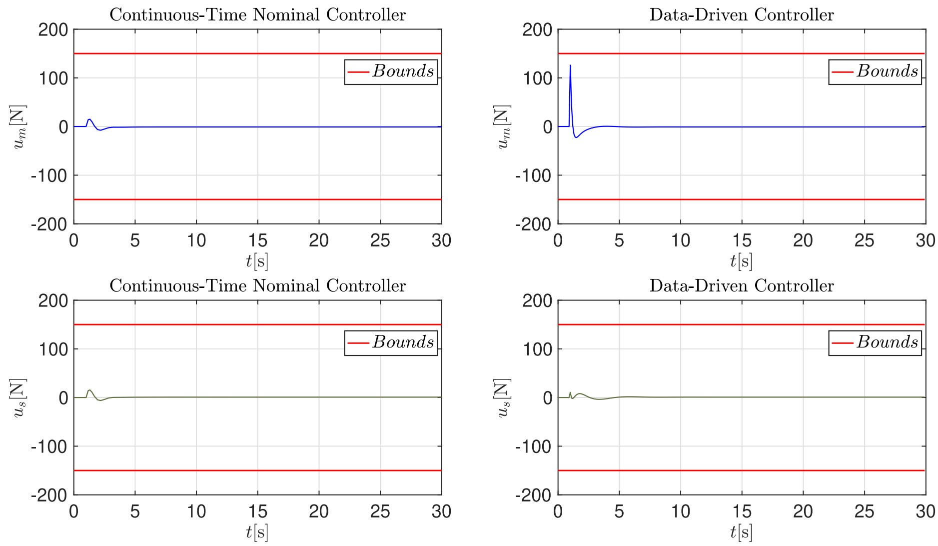

Fig. 4. Variation of control inputs for Case Study 1.

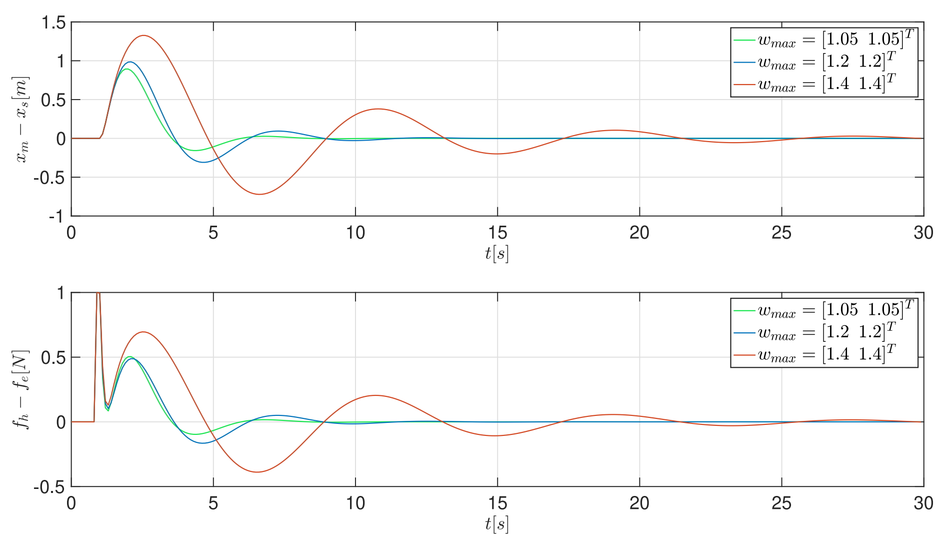

Fig. 5. Variation of tracking errors for different bounds on exogenous forces.

the pre-defined physical limits. Fig. 4 shows the control inputs in the corresponding controllers. It can be observed from the figure that both controllers operate within input limits. Moreover, the DDC produces more than ten times higher control force than the nominal case on the master side (u_m) and a lower control force on the slave side (u_s). Another significant analysis can be drawn regarding the performance of the proposed controller under different input bounds. The proposed controller's reference and force tracking performances under different exogenous force bounds are demonstrated through error signals in Fig. 5. As the figure shows, designing the controller in a narrow bound of exogenous forces improves the reference and force tracking performance. Besides, disturbances exceeding the bounds on exogenous forces $w_{max} = \begin{bmatrix} 1.2 & 1.2 \end{bmatrix}^T$, affect the system unfavourably. Finally, Fig. 6 shows the proposed controller's reference and force tracking performance under different control input bounds. While the reference and force tracking performance does not present any remarkable changes, slightly better performance is seen for control forces above 100[N].

The simulation results in Case Study 1 show that the proposed method performs similarly to the nominal case controller but produces higher controller forces and $\mathcal{H}_\infty$ norm. Notably, the proposed controller does not require knowledge of A and B_u, unlike the nominal control, yet it still performs successfully. These simulation results demonstrate the value of the proposed method.

5.2. Case study 2

This case study aims to demonstrate that our proposed DD controller can effectively learn during training. To evaluate the performance of our controller, we conducted ten simulations with the same physical bounds, parameters, and training duration, N. We also compared our results to those obtained using Berberich et al.'s method [12] with the same training duration, N. The results of these simulations are shown in Table 2 in terms of γ and % error values. Note that γ_t represents the target γ. As summarized in Table 2, our method resulted in almost 30% lower percentage error compared to Berberich et al.'s method, which is presented in the literature as another DDC.

In addition to converging to the exact $\mathcal{H}_\infty$ gain, γ_t, it is essential to achieve a high degree of repeatability or consistent performance on reference/force tracking in every experiment.

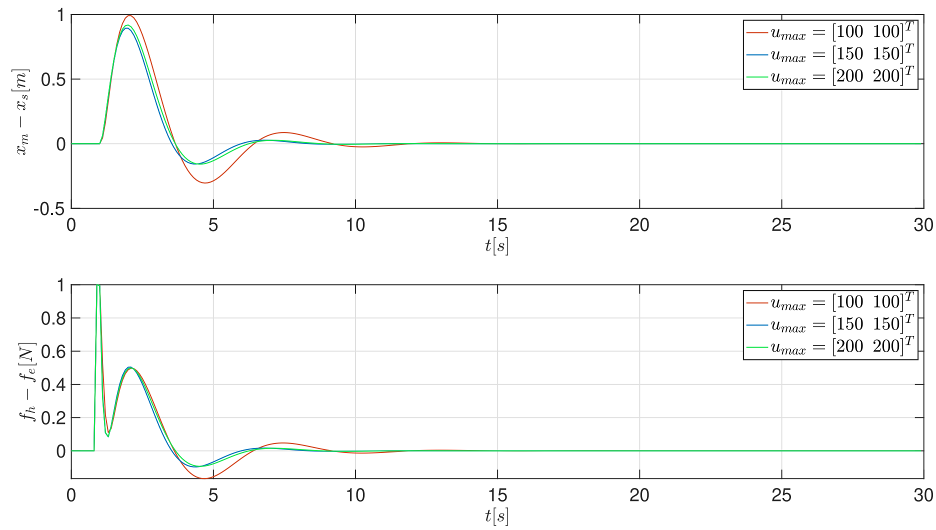

Fig. 6. Variation of error signals for different bounds on control Inputs.

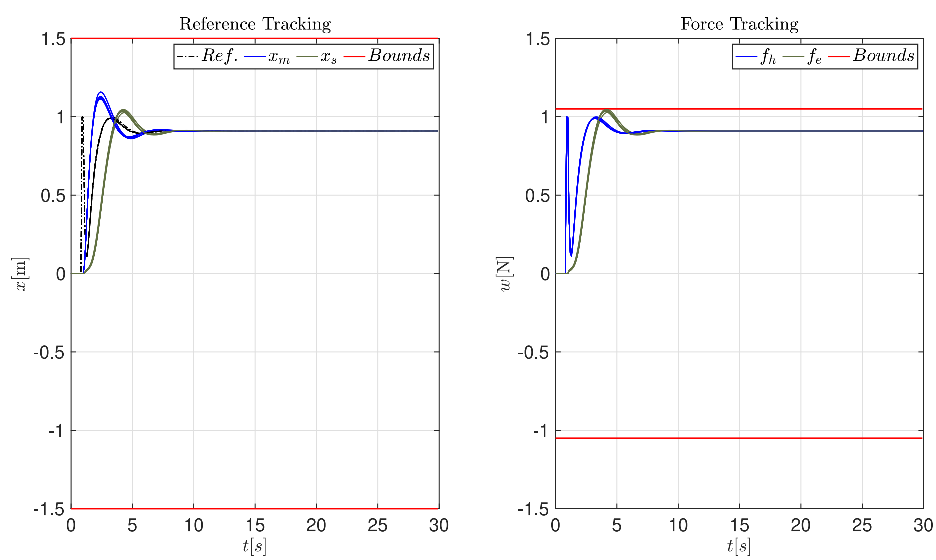

Fig. 7. Reference/Force tracking for Case Study 2.

Table 2
A comparison study in terms of learning performance.

Experiment	Our Method ($\gamma_t = 7.5$, $N = 20$)		Berberich et al.'s method [12] ($\gamma_t = 2.4$, $N = 20$)	
	γ	% Error	γ	% Error
1	7.44	0.8%	2.24	6.67%
2	7.19	4.13%	2.24	6.67%
3	7.49	0.13%	2.61	8.75%
4	8.16	8.8%	2.26	5.83%
5	7.84	4.53%	2.40	0.00%
6	7.79	3.86%	2.33	2.92%
7	7.56	0.8%	2.28	5.00%
8	7.72	2.93%	2.41	0.42%
9	7.70	2.67%	2.54	5.83%
10	7.73	3.06%	2.35	2.08%
	Average % Error → 3.17%		Average % Error → 4.42%	

These experiments must also be conducted under bounded conditions. To demonstrate repeatability, Figs. 7 and 8 show the reference/force tracking and controller forces for ten successive experiments, where the results are plotted on the same figure.

Fig. 7 shows that the proposed controller performs similarly in every experiment. Moreover, all bounded trajectories never exceed the specified state limits. Likewise, Fig. 8 demonstrates that control inputs always stay within limits, and the controller produces similar control input values.

The identification-free design method allows the synthesis of an easily implementable controller. While this method does not require any information about the system matrices, A and B_u, it can still provide successful reference and force tracking performance compared to the CT $\mathcal{H}_\infty$ controller, which does require full system knowledge. In addition, the state, disturbance, and control input values are all within the pre-defined physical bounds. Our DDC also outperforms the related, well-accepted controller from the literature by achieving almost 30% lower percentage error between γ and γ_t. These simulation studies demonstrate that the proposed design method can generate a less conservative controller that performs well under physical constraints and norm-bounded disturbances.

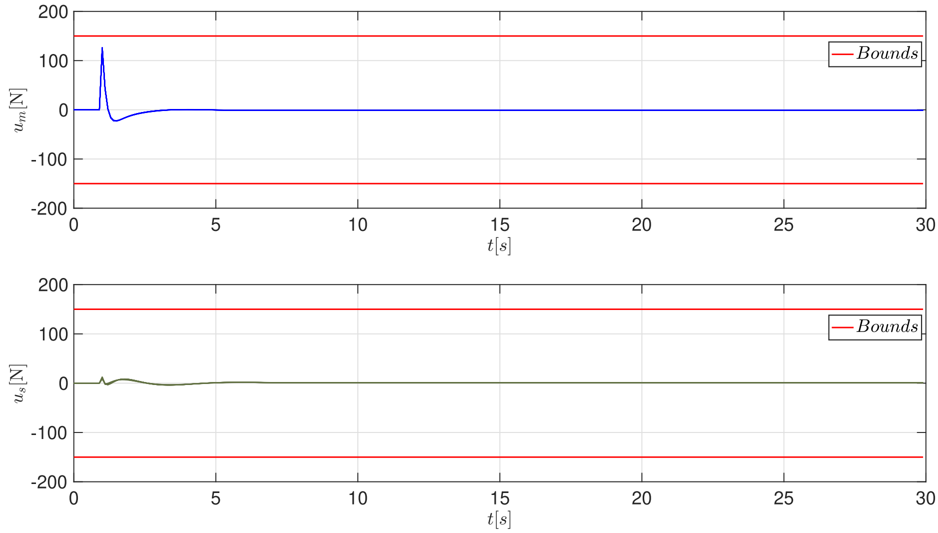

Fig. 8. Control inputs for Case Study 2.

6. Discussions and conclusions

We considered the design of an LMI-based DD $\mathcal{H}_\infty$ controller for discrete-time LTI systems having physical constraints on inputs and states and subject to norm-bounded disturbance. The proposed DD controller method does not require the knowledge of system matrices except the disturbance input matrix and the performance output matrix determined by the user in the controller design process.

The developed methodology involves expressing the system matrices as a function of control input and state measurements. To obtain a less conservative and easily implementable DD synthesis, we used FBSPM, which employs a convex hull relaxation approach. However, other relaxation techniques, such as Pólya's approach or Sum of Squares (SoS), could also be utilized. This work offers a novel, identification-free DD design methodology that is easy to implement and has minimal conservatism, relying on LMI conditions. The optimization-based approach used in the proposed state-feedback $\mathcal{H}_\infty$ controller ensures the best $\mathcal{H}_\infty$ performance under the physical constraints on states and inputs.

To evaluate the effectiveness of our proposed method, we used a bilateral teleoperation system for reference tracking and conducted two case studies through simulations. In the first case, we compared our approach to a nominal CT $\mathcal{H}_\infty$ controller that considers saturation nonlinearity in the control inputs. Our method provided a practical and acceptable control strategy in this case. In the second case, we compared our method to a recently developed DD controller from the literature and found that our proposed controller had superior learning performance. Additionally, our proposed controller was able to maintain the state and input variables within the desired bounds. As a potential area for future research, it would be interesting to investigate the possibility of extending our proposed DD $\mathcal{H}_\infty$ control strategy to achieve robustness against uncertainties in system parameters.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix. Continuous-time nominal controller having saturated actuator

Using the SF control law $u(t) = Kx(t)$ along with (46), one can easily obtain a CT nominal CL system model as follows:

$$\mathcal{H}_{cl_{ct}} \left\{ \begin{bmatrix} \dot{x}(t) \\ \hline z(t) \end{bmatrix} = \left[\begin{array}{c|c} \mathcal{A}_{cl_{ct}} & \mathcal{B}_{cl_{ct}} \\ \hline \mathcal{C}_{cl_{ct}} & \mathcal{D}_{cl_{ct}} \end{array} \right] \begin{bmatrix} x(t) \\ \hline w(t) \end{bmatrix}, \right. \tag{A.1}$$

where

$$\mathcal{A}_{cl_{ct}} = A_{ct} + B_{u_{ct}} K,\ \mathcal{B}_{cl_{ct}} = B_{w_{ct}},\ \mathcal{C}_{cl_{ct}} = C_{ct} + D_{ct} K,\ \mathcal{D}_{cl_{ct}} = D_{w_{ct}}. \tag{A.2}$$

Lemma 7 (*[21]*)**.** *For a given $\gamma > 0$, the CL system (A.1) is asymptotically stable and satisfies $\|\mathcal{H}_{cl_{ct}}\|_\infty < \gamma$ for all $t \geq 0$ with a control law $u(t) = SF^{-1}x(t)$ if and only if, there exist an appropriate dimensioned matrix S and $F = F^T \succ 0$ such that*

$$\begin{bmatrix} A_{ct}F + FA_{ct}^T + B_{u_{ct}}S + S^T B_{u_{ct}}^T & B_{w_{ct}} & FC_{ct}^T + S^T D_{ct}^T \\ \star & -\gamma I & D_{w_{ct}}^T \\ \star & \star & -\gamma I \end{bmatrix} \prec 0. \tag{A.3}$$

Lemma 8 (*[30]*)**.** *The CL system (A.1) is asymptotically stable and satisfies $\|\mathcal{H}_{cl_{ct}}\|_\infty < \gamma$ for all $t \geq 0$ and $u^T u \preceq u_{max}^2$ if there exist appropriately dimensioned matrices S and $F = F^T$ that satisfy*

$$\begin{bmatrix} F & S^T \\ \star & u_{max}^2 I \end{bmatrix} \succ 0 \tag{A.4}$$

and (A.3). Then, a CT nominal, static SF controller for an LTI system having saturated actuators is obtained by $u(t) = SF^{-1}x(t)$.

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