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Some properties of generalized complex space forms

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Abstract. In the present paper, we determine the holomorphic curvature tensor of generalized complex space forms and study some properties of this tensor in generalized complex space forms. Moreover, we present results on generalized complex space forms satisfying curvature identities named Walker type identities.

Keywords: Generalized complex space form, Holomorphic curvature tensor, Walker type identity.

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INTRODUCTION

In 1989, Z. Olszak has worked on the existence of a generalized complex space form [1]. In [2], U.C. De and A. Sarkar studied the nature of a generalized Sasakian space form under some conditions regarding projective curvature tensor. They also studied Sasakian space forms with vanishing quasi-conformal curvature tensor and investigated quasi-conformal flat generalized Sasakian space forms, Ricci-symmetric and Ricci semisymmetric generalized Sasakian space forms [3]. M.C. Bharathi and C. S. Bagewadi [4] extended the study to W_2 curvature, conharmonic and concircular curvature tensors on generalized complex space forms. Moreover, P. Mutlu studied some curvature properties of generalized complex space forms [5].

On the other hand, M. Prvanović [6] has introduced a tensor of Kaehler type for an almost Hermitian manifold, this tensor reduces to the Riemannian curvature tensor R in an almost Kaehler manifold. Then M. Prvanović gave some properties about such tensor [7, 8, etc.].

Motivated by these ideas, in the present paper, we introduce the holomorphic curvature tensor for generalized complex space forms and study some properties of this tensor in generalized complex space forms. Moreover, we study generalized complex space forms satisfying curvature identities named Walker type identities.

Results

A Kaehler manifold is an even dimensional manifold M^m , where m=2n with a complex structure J and a positive definite metric g which satisfies the following conditions [9]

$$J^2 = -Id. \quad , \quad g(JX, JY) = g(X, Y) \quad and \quad \nabla J = 0, \tag{1}$$

where ∇ denotes the covariant derivative with respect to Levi-Civita connection.

Let (M, g, J) be a Kaehler manifold with constant holomorphic sectional curvature $K(X \wedge JX) = c$, then it is said to be a complex space form and it is well known that its curvature tensor satisfies the equation

$$\begin{aligned} R(X, Y)Z &= \frac{c}{4} \{g(Y, Z)X - g(X, Z)Y + g(X, JZ)JY - g(Y, JZ)JX \\ &+ 2g(X, JY)JZ\} \end{aligned} \tag{2}$$

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for any vector fields X ,Y, Z on M.

More generally, if the curvature tensor of an almost Hermitian manifold M satisfies

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2\{g(X,JZ)JY - g(Y,JZ)JX + 2g(X,JY)JZ\}, \end{aligned} \tag{3}$$

for differentiable functions f_1, f_2 on M, then M is said to be a generalized complex space form M ([10], [11]).

In an m-dimensional generalized complex space form $M(f_1, f_2)$ we have

$$S(X,Y) = [(m-1)f_1 + 3f_2]\ g(X,Y),$$

$$QX = [(m-1)f_1 + 3f_2]\ X,$$

$$\kappa = m[(m-1)f_1 + 3f_2],$$

where S is the Ricci tensor, Q is the Ricci operator and κ is the scalar curvature of $M(f_1, f_2)$.

If we put $R^*(X,Y,Z,W) = R(X,Y,JZ,JW)$ then the second Ricci tensor, S^*, is the Ricci tensor associated to the tensor R^*, i.e.,

$$S^*(X,Y) = \sum_i R^*(e_i, X, Y, e_i) = \sum_i R(e_i, X, JY, Je_i),$$

where $\{e_i\}$ is an orthonormal basis. The second scalar curvature is defined by

$$\kappa^* = \sum_i S^*(e_i, e_i).$$

The Riemannian curvature tensor R of a Kaehler manifold satisfies the condition

$$R(X,Y,JZ,JW) = R(X,Y,Z,W). \tag{4}$$

The relation (4) is the Kaehler identity. If $\nabla J \neq 0$, (4) does not hold in general. Nevertheless, there exist for any almost Hermitian manifold the algebraic curvature tensor, satisfying the condition of type (4). It is ([6], [12]):

$$\begin{aligned} (HR)(X,Y,Z,W) &= \frac{1}{16}\Big[3\Big[R(X,Y,Z,W) + R(JX,JY,Z,W) + R(X,Y,JZ,JW) \\ &+ R(JX,JY,JZ,JW)\Big] - R(X,Z,JW,JY) - R(JX,JZ,W,Y) \\ &- R(X,W,JY,JZ) - R(JX,JW,Y,Z) + R(JX,Z,JW,Y) \\ &+ R(X,JZ,W,JY) + R(JX,W,Y,JZ) + R(X,JW,JY,Z)\Big]. \end{aligned} \tag{5}$$

The tensor (5) is said to be the holomorphic curvature tensor in an almost Hermitian manifold.

Theorem 1 *The holomorphic curvature tensor of an m-dimensional* $(m > 2)$ *generalized complex space form* $M(f_1, f_2)$ *has the form*

$$\begin{aligned} (HR)(X,Y)Z &= R(X,Y)Z - \frac{\kappa - \kappa^*}{4m(m-2)}\Big[3\Big[g(Y,Z)X - g(X,Z)Y\Big] \\ &- \Big[g(X,JZ)JY - g(Y,JZ)JX + 2g(X,JY)JZ\Big]\Big]. \end{aligned} \tag{6}$$

Theorem 2 *Let* $M(f_1, f_2)$ *be an m-dimensional* $(m > 2)$ *generalized complex space form such that* κ *and* κ^* *are constants. If* $HR = 0$ *, then* $M(f_1, f_2)$ *is Einstein.*

For a (0,k)-tensor field T, $k \geq 1$, the (0,k+2)-tensor fields $R \cdot T$ and $Q(g, T)$ are given by

$$\begin{aligned} (R \cdot T)(X_1, ..., X_k; X, Y) &= -T(R(X, Y)X_1, X_2, ..., X_k) \\ &- ... - T(X_1, X_2, ..., X_{k-1}, R(X, Y)X_k), \end{aligned}$$

$$\begin{aligned} Q(g, T)(X_1, ..., X_k; X, Y) &= -T((X \wedge_g Y)X_1, X_2, ..., X_k) \\ &- ... - T(X_1, X_2, ..., X_{k-1}, (X \wedge_g Y)X_k), \end{aligned}$$

respectively. (see, e.g., [13-16]).

Theorem 3 *Let $M(f_1, f_2)$ be an m-dimensional $(m > 2)$ generalized complex space form. Then we have*

$$R \cdot HR = R \cdot R + \frac{\kappa - \kappa^*}{4m(m-2)} E \tag{7}$$

and

$$HR \cdot R = R \cdot R - \frac{\kappa - \kappa^*}{4m(m-2)} [3Q(g, R) - \bar{E}], \tag{8}$$

where

$$\begin{aligned} E_{hijklm} &= J_{ij}(A_{hklm} - A_{khlm}) + J_{hk}(A_{ijlm} - A_{jilm}) \\ &+ J_{ik}(A_{jhlm} - A_{hjlm}) + J_{hj}(A_{kilm} - A_{iklm}) \\ &+ 2J_{jk}(A_{ihlm} - A_{hilm}) + 2J_{hi}(A_{kjlm} - A_{jklm}), \end{aligned}$$

$$\begin{aligned} \bar{E}_{hijklm} &= \Big(J_{lh}A_{mijk} + J_{hm}A_{lijk} + J_{il}A_{mhjk} - J_{im}A_{lhjk} \\ &- J_{jl}A_{mkhi} + J_{jm}A_{lkhi} + J_{kl}A_{mjhi} - J_{km}A_{ljhi}\Big) \\ &+ 2J_{lm}(A_{hijk} - A_{ihjk} + A_{jkhi} - A_{kjhi}) \end{aligned}$$

and $A_{ijlm} = J_i^s R_{sjlm}$.

Theorem 4 *Let $M(f_1, f_2)$ be an m-dimensional $(m > 2)$ generalized complex space form. Then the following three identities are equivalent to each other :*

$$i) \sum_{(X_1,X_2)(X_3,X_4)(X,Y)} (R \cdot HR)(X_1, X_2, X_3, X_4; X, Y) = 0, \tag{9}$$

$$ii) \sum_{(X_1,X_2)(X_3,X_4)(X,Y)} (HR \cdot R)(X_1, X_2, X_3, X_4; X, Y) = 0, \tag{10}$$

$$iii) \sum_{(X_1,X_2)(X_3,X_4)(X,Y)} (R \cdot HR - HR \cdot R)(X_1, X_2, X_3, X_4; X, Y) = 0 \tag{11}$$

on $M(f_1, f_2)$.

Theorem 5 *Let $M(f_1, f_2)$ be an m-dimensional $(m > 2)$ generalized complex space form. Then the Walker type identities (9)-(11) hold on $M(f_1, f_2)$.*

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